

1 CONTRACTIBLE STABILITY SPACES AND FAITHFUL 2 BRAID GROUP ACTIONS

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ABSTRACT. We prove that any ‘finite-type’ component of a stability space of a triangulated category is contractible. The motivating example of such a component is the stability space of the Calabi–Yau- N category $\mathcal{D}(\Gamma_N Q)$ associated to an ADE Dynkin quiver. In addition to showing that this is contractible we prove that the braid group $\mathrm{Br}(Q)$ acts freely upon it by spherical twists, in particular that the spherical twist group $\mathrm{Br}(\Gamma_N Q)$ is isomorphic to $\mathrm{Br}(Q)$. This generalises Brav–Thomas’ result for the $N = 2$ case. Other classes of triangulated categories with finite-type components in their stability spaces include locally-finite triangulated categories with finite rank Grothendieck group and discrete derived categories of finite global dimension.

Key words: Stability conditions, Calabi–Yau categories, spherical twists, braid groups

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12 1. INTRODUCTION

13 **1.1. Stability conditions.** Spaces of stability conditions on a triangulated
14 category were introduced in [12], inspired by the work of Michael Douglas on
15 stability of D-branes in string theory. The construction associates a space
16 $\mathrm{Stab}(\mathcal{C})$ of stability conditions to each triangulated category \mathcal{C} . A stability
17 condition $\sigma \in \mathrm{Stab}(\mathcal{C})$ consists of a *slicing* — for each $\varphi \in \mathbb{R}$ an abelian
18 subcategory $\mathcal{P}_\sigma(\varphi)$ of *semistable objects of phase* φ such that each object of
19 \mathcal{C} can be expressed as an iterated extension of semistable objects — and a
20 *central charge* $Z: K\mathcal{C} \rightarrow \mathbb{C}$ mapping the Grothendieck group $K\mathcal{C}$ linearly
21 to \mathbb{C} . The slicing and charge obey a short list of axioms. The miracle is
22 that the space $\mathrm{Stab}(\mathcal{C})$ of stability conditions is a complex manifold, locally
23 modelled on a linear subspace of $\mathrm{Hom}(K\mathcal{C}, \mathbb{C})$ [12, Theorem 1.2]. It carries

commuting actions of \mathbb{C} , acting by rotating phases and rescaling masses, and of the automorphism group $\text{Aut}(\mathcal{C})$.

Whilst a number of examples of spaces of stability conditions are known, it is in general difficult to compute $\text{Stab}(\mathcal{C})$. It is widely believed that spaces of stability conditions are contractible, and this has been verified in certain examples. We give the first proof of contractibility for certain general classes of triangulated categories satisfying (strong) finiteness conditions.

Our strategy is to identify general conditions under which there are no ‘complicated’ stability conditions. One measure of the complexity of a stability condition σ is the phase distribution, i.e. the set $\{\varphi \in \mathbb{R} \mid \mathcal{P}_\sigma(\varphi) \neq \emptyset\}$ of phases for which there is a non-zero semistable object. A good heuristic is that a stability condition with a dense phase distribution is complicated, whereas one with a discrete phase distribution is much less so — see [21] for a precise illustration of this principle.

Another measure of complexity is provided by the properties of the heart of the stability condition σ . This is the full extension-closed subcategory $\mathcal{P}_\sigma(0, 1]$ generated by the semistable objects with phases in the interval $(0, 1]$. It is the heart of a bounded t-structure on \mathcal{C} and so in particular is an abelian category. From this perspective the ‘simplest’ stability conditions are those whose heart is Artinian and Noetherian with finitely many isomorphism class of simple objects; we call these *algebraic* stability conditions.

These two measures of complexity are related: if there is at least one algebraic stability condition then the union $\mathbb{C} \cdot \text{Stab}_{\text{alg}}(\mathcal{C})$ of orbits of algebraic stability conditions under the \mathbb{C} -action is the set of stability conditions whose phase distribution is not dense.

We show that the subset $\text{Stab}_{\text{alg}}(\mathcal{C})$ is stratified by real submanifolds, each consisting of stability conditions for which the heart is fixed and a given subset of its simple objects have integral phases. Each of these strata is contractible, so the topology of $\text{Stab}_{\text{alg}}(\mathcal{C})$ is governed by the combinatorics of adjacencies of strata. It is well-known that as one moves in $\text{Stab}(\mathcal{C})$ the associated heart changes by Happel–Reiten–Smalø tilts. The combinatorics of tilting is encoded in the poset $\text{Tilt}(\mathcal{C})$ of t-structures on \mathcal{C} with relation $\mathcal{D} \leq \mathcal{E} \iff$ there is a finite sequence of (left) tilts from \mathcal{D} to \mathcal{E} . Components of this poset are in bijection with components of $\text{Stab}_{\text{alg}}(\mathcal{C})$. Corollary 3.13 describes the precise relationship between $\text{Tilt}(\mathcal{C})$ and the stratification of $\text{Stab}_{\text{alg}}(\mathcal{C})$. Using this connection we obtain our main theorem:

Theorem A (Lemma 4.3 and Theorem 4.9). *Suppose each algebraic t-structure in some component of $\text{Tilt}(\mathcal{C})$ has only finitely many tilts, all of which are algebraic. Then the corresponding component of $\text{Stab}_{\text{alg}}(\mathcal{C})$ is actually a component of $\text{Stab}(\mathcal{C})$, and moreover is contractible.*

We say that a component satisfying the conditions of the theorem has *finite-type*. The phase distribution of any stability condition in a finite-type component is discrete. It seems plausible that the converse is true, i.e. that any component of $\text{Stab}(\mathcal{C})$ consisting entirely of stability conditions with discrete phase distribution is a finite-type component, but we have not been able to prove this. There are several interesting classes of examples of finite-type components. We show that if \mathcal{C} is

- 71 • a locally-finite triangulated category with finite rank Grothendieck
72 group ([35], see Section 4.2), then any component of $\text{Stab}(\mathcal{C})$ is of
73 finite-type;
- 74 • a discrete derived category of finite global dimension (see Section 4.3),
75 then $\text{Stab}(\mathcal{C})$ consists of a single finite-type component;
- 76 • the bounded derived category $\mathcal{D}(\Gamma_N Q)$ of finite-dimensional repre-
77 sentations of the Calabi–Yau- N Ginzburg algebra of a Dynkin quiver
78 Q , for any $N \geq 2$, then the space of stability conditions has finite-
79 type.

80 The bounded derived category $\mathcal{D}(Q)$ of a Dynkin quiver Q is both locally-
81 finite and discrete, and the first two classes can be seen as different ways
82 to generalise from these basic examples. Perhaps surprisingly, until now the
83 space of stability conditions on $\mathcal{D}(Q)$ was only known to be contractible
84 for Q of type A_1 or A_2 , although it was known by [43] that it was simply-
85 connected.

86 Similarly, for discrete derived categories contractibility was known before
87 only for the simplest case, which was treated in [52]. The description of
88 the stratification of $\text{Stab}(\mathcal{D})$ for \mathcal{D} a discrete derived category, from which
89 contractibility follows, was obtained independently, and simultaneously with
90 our results, in [19]. They use an alternative algebraic interpretation of the
91 combinatorics of the stratification in terms of silting subcategories and silt-
92 ing mutation.

93 The third class of examples has been the most intensively studied. The
94 space of stability conditions $\text{Stab}(\Gamma_N Q)$ has been identified as a complex
95 space in various cases, in each of which it is known to be contractible. The
96 connectedness of $\text{Stab}(\Gamma_N Q)$ is proven by [1] recently for the Dynkin case.
97 For $N = 2$ and Q a quiver of type A it was first studied in [49], where
98 the stability space was shown to be the universal cover of a configuration
99 space of points in the complex plane. Using different methods [14] identified
100 $\text{Stab}(\Gamma_2 Q)$ for any Dynkin quiver Q as a covering space using a geometric
101 description in terms of Kleinian singularities. Later [11], see also [43], showed
102 that it was the *universal* cover in all these cases. When the underlying
103 Dynkin diagram of Q is A_n , [26] shows that $\text{Stab}(\Gamma_N Q)$ is the universal cover
104 of the space of degree $n+1$ polynomials $p_n(z)$ with simple zeros. The central
105 charges are constructed as periods of the quadratic differential $p_n(z)^{N-2} dz^{\otimes 2}$
106 on \mathbb{P}^1 , using the technique of [16]. The $N = 3$ case of this result was
107 obtained previously in [48]. The A_2 case for arbitrary N , including $N =$
108 ∞ which corresponds to stability conditions on $\mathcal{D}(A_2)$, was treated in [15]
109 using different methods. Besides, [27] showed that $\text{Stab}(\Gamma_2 Q)$ is connected,
110 and also that the stability space of the affine counterpart is connected and
111 simply-connected. Our methods do not apply to this latter case. Finally,
112 [42] proved the contractibility of the principal component of $\text{Stab}(\Gamma_3 Q)$ for
113 any affine A type quivers.

114 Although there are several interesting classes of examples, the finiteness
115 condition required for our theorem is strong. For instance it is not satisfied
116 by tame representation type quivers such as the Kronecker quiver. Different
117 methods will probably be required in these cases, because the stratification
118 of the space of algebraic stability conditions fails to be locally-finite and

closure-finite, and so is much harder to understand and utilise. Examples of alternative methods for proving the contractibility of the space of stability conditions on $\mathcal{D}(Q)$ can be found in [38] for the case of the Kronecker quiver, and [22] for the case of the acyclic triangular quiver.

1.2. Representations of braid groups. One can associate a braid group $\text{Br}(Q)$ to an acyclic quiver Q — it is defined by having a generator for each vertex, with a braid relation $aba = bab$ between generators whenever the corresponding vertices are connected by an arrow, and a commuting relation $ab = ba$ whenever they are not. For example, the braid group of the A_n quiver is the standard braid group on $n + 1$ strands.

This braid group acts on $\mathcal{D}(\Gamma_N Q)$ by spherical twists. The image of $\text{Br}(Q)$ in the group of automorphisms is the Seidel–Thomas braid group $\text{Br}(\Gamma_N Q)$. Its properties are closely connected to the topology of $\text{Stab}(\Gamma_N Q)$, in particular $\text{Stab}(\Gamma_N Q)$ is simply-connected whenever the Seidel–Thomas braid action on it is faithful.

The Seidel–Thomas braid group originated in the study of Kontsevich’s homological mirror symmetry. On the symplectic side, Khovanov–Seidel [32] showed that when Q has type A the category $\mathcal{D}(\Gamma_N Q)$ can be realised as a subcategory of the derived Fukaya category of the Milnor fibre of a simple singularity of type A . Here $\text{Br}(Q)$ acts as (higher) Dehn twists along Lagrangian spheres, and they proved this action is faithful. On the algebraic geometry side, Seidel–Thomas [46] studied the mirror counterpart of [32]; here $\mathcal{D}(\Gamma_N Q)$ can be realised as a subcategory of the bounded derived category of coherent sheaves of the mirror variety.

The proofs of faithfulness of the braid group action by Khovanov–Seidel–Thomas ([32, 46]) depend on the existence of a faithful geometric representation of the braid group in the mapping class group of a surface. Such faithful actions are known to exist by Birman–Hilden [8] when Q has type A , and by Perron–Vannier [40] when Q has type D . Surprisingly, Wajsbury [51] showed that there is no such faithful geometric representation of the braid group of type E , so this method of proof cannot be generalised to all Dynkin quivers. A different approach, relying on the *Garside structure* on the braid group $\text{Br}(Q)$, was used by Brav–H. Thomas [11] to prove that the braid group action on $\mathcal{D}(\Gamma_2 Q)$ is faithful for all Dynkin quivers Q . The $N = 2$ case is the simplest because $\text{Br}(Q)$ acts transitively on the tilting poset $\text{Tilt}(\Gamma_N Q)$; this is not so for $N \geq 3$. Nevertheless, we are able to ‘bootstrap’ from the $N = 2$ case to prove:

Theorem B (Corollaries 5.1, 6.12, and 6.14). *For any Dynkin quiver Q and any $N \geq 2$ the action of $\text{Br}(Q)$ on $\mathcal{D}(\Gamma_N Q)$ is faithful, and the induced action on $\text{Stab}(\Gamma_N Q)$ is free. Moreover, $\text{Stab}(\Gamma_N Q)$ is contractible and the finite-dimensional complex manifold $\text{Stab}(\Gamma_N Q) / \text{Br}(Q)$ is a model for the classifying space of $\text{Br}(Q)$.*

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2. PRELIMINARIES

Throughout the paper, k is a fixed (not necessarily algebraically-closed)
 field. The Grothendieck group of an abelian, or triangulated, category \mathcal{C} is
 denoted by $K\mathcal{C}$.

The bounded derived category of the path algebra kQ of a quiver Q
 is denoted $\mathcal{D}(Q)$ and the bounded derived category of finite-dimensional
 representations of the Calabi–Yau– N Ginzburg algebra of a Dynkin quiver Q ,
 for $N \geq 2$, is denoted $\mathcal{D}(\Gamma_N Q)$. The bounded derived category of coherent
 sheaves on a variety X over k is denoted $\mathcal{D}(X)$. The spaces of locally-finite
 stability conditions on these triangulated categories are denoted by $\text{Stab}(Q)$,
 by $\text{Stab}(\Gamma_N Q)$ and by $\text{Stab}(X)$ respectively.

2.1. Posets. Let P be a poset. We denote the closed interval

$$\{r \in P : p \leq r \leq q\}$$

by $[p, q]$, and similarly use the notation $(-\infty, p]$ and $[p, \infty)$ for bounded
 above and below intervals. A poset is *bounded* if it has both a minimal
 and a maximal element. A *chain* of length k in a poset P is a sequence
 $p_0 < \dots < p_k$ of elements. One says q *covers* p if $p < q$ and there does not
 exist $r \in P$ with $p < r < q$. A chain $p_0 < \dots < p_k$ is said to be *unrefinable*
 if p_i covers p_{i-1} for each $i = 2, \dots, k$. A *maximal* chain is an unrefinable
 chain in which p_i is a minimal element and p_k a maximal one. A poset is
pure if all maximal chains have the same length; the common length is then
 called the *length* of the poset.

A poset determines a simplicial set whose k -simplices are the non-strict
 chains $p_0 \leq \dots \leq p_k$ in P . The *classifying space* BP of P is the geometric
 realisation of this simplicial set. If we view P as a category with objects
 the elements and a (unique) morphism $p \rightarrow q$ whenever $p \leq q$, the above
 simplicial set is the *nerve*, and BP is the classifying space of the category
 in the usual sense, see [44, §2].

Elements p and q are said to be in the same *component* of P if there is a
 sequence of elements $p = p_0, p_1, \dots, p_k = q$ such that either $p_i \leq p_{i+1}$ or $p_i \geq$
 p_{i+1} for each $i = 0, \dots, k-1$; equivalently if the 0-simplices corresponding
 to p and q are in the same component of the classifying space BP .

The classifying space is a rather crude invariant of P . For example, there
 is a homeomorphism $BP \cong BP^{\text{op}}$, and if each finite set of elements has an
 upper bound (or a lower bound) then the classifying space BP is contractible
 by [44, Corollary 2] since P , considered as a category, is filtered.

2.2. t-structures. We fix some notation. Let \mathcal{C} be an additive category.
 We write $c \in \mathcal{C}$ to mean c is an object of \mathcal{C} . We will use the term *subcategory*
 to mean strict, full subcategory. When S is a subcategory we write \mathcal{S}^\perp for
 the subcategory on the objects

$$\{c \in \mathcal{C} : \text{Hom}_{\mathcal{C}}(s, c) = 0 \ \forall s \in \mathcal{S}\}$$

208 and similarly ${}^\perp\mathcal{S}$ for $\{c \in \mathcal{C} : \text{Hom}_{\mathcal{C}}(c, s) = 0 \ \forall s \in \mathcal{S}\}$. When \mathcal{A} and \mathcal{B} are
 209 subcategories of \mathcal{C} we write $\mathcal{A} \cap \mathcal{B}$ for the subcategory on objects which lie
 210 in both \mathcal{A} and \mathcal{B} .

211 Suppose \mathcal{C} is triangulated, with shift functor $[1]$. Exact triangles in \mathcal{C} will
 212 be denoted either by $a \rightarrow b \rightarrow c \rightarrow a[1]$ or by a diagram

$$\begin{array}{ccc} a & \xrightarrow{\quad} & b \\ & \swarrow \text{dotted} & \searrow \\ & c & \end{array}$$

214 where the dotted arrow denotes a map $c \rightarrow a[1]$. We will always assume
 215 that \mathcal{C} is essentially small so that isomorphism classes of objects form a set.
 216 Given sets E_i of objects for $i \in I$ let $\langle E_i \mid i \in I \rangle$ denote the ext-closed
 217 subcategory generated by objects isomorphic to an element in some E_i . We
 218 will use the same notation when the E_i are subcategories of \mathcal{C} .

219 **Definition 2.1.** A *t-structure* on a triangulated category \mathcal{C} is an ordered
 220 pair $\mathcal{D} = (\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 1})$ of subcategories, satisfying:

- 221 (1) $\mathcal{D}^{\leq 0}[1] \subseteq \mathcal{D}^{\leq 0}$ and $\mathcal{D}^{\geq 1}[-1] \subseteq \mathcal{D}^{\geq 1}$;
- 222 (2) $\text{Hom}_{\mathcal{C}}(d, d') = 0$ whenever $d \in \mathcal{D}^{\leq 0}$ and $d' \in \mathcal{D}^{\geq 1}$;
- 223 (3) for any $c \in \mathcal{C}$ there is an exact triangle $d \rightarrow c \rightarrow d' \rightarrow d[1]$ with
 224 $d \in \mathcal{D}^{\leq 0}$ and $d' \in \mathcal{D}^{\geq 1}$.

225 We write $\mathcal{D}^{\leq n}$ to denote the shift $\mathcal{D}^{\leq 0}[-n]$, and so on. The subcategory $\mathcal{D}^{\leq 0}$
 226 is called the *aisle* and $\mathcal{D}^{\geq 0}$ the *co-aisle* of the t-structure. The intersection
 227 $\mathcal{D}^0 = \mathcal{D}^{\geq 0} \cap \mathcal{D}^{\leq 0}$ of the aisle and co-aisle is an abelian category known as
 228 the *heart* of the t-structure — see [6, Théorème 1.3.6] or [28, §10.1].

229 The exact triangle $d \rightarrow c \rightarrow d' \rightarrow d[1]$ is unique up to isomorphism. The
 230 first term determines a right adjoint to the inclusion $\mathcal{D}^{\leq 0} \hookrightarrow \mathcal{C}$ and the last
 231 term a left adjoint to the inclusion $\mathcal{D}^{\geq 1} \hookrightarrow \mathcal{C}$.

232 A t-structure \mathcal{D} is *bounded* if any object of \mathcal{C} lies in $\mathcal{D}^{\geq -n} \cap \mathcal{D}^{\leq n}$ for some
 233 $n \in \mathbb{N}$.

234 *Henceforth, we will assume that all t-structures are bounded.*

235 This has three important consequences. Firstly, a bounded t-structure is
 236 completely determined by its heart; the aisle is recovered as

$$\mathcal{D}^{\leq 0} = \langle \mathcal{D}^0, \mathcal{D}^{-1}, \mathcal{D}^{-2}, \dots \rangle.$$

237 Secondly, the inclusion $\mathcal{D}^0 \hookrightarrow \mathcal{C}$ induces an isomorphism $K\mathcal{D}^0 \cong K\mathcal{C}$ of
 238 Grothendieck groups. Thirdly, if $\mathcal{D}^0 \subseteq \mathcal{E}^0$ are hearts of bounded t-structures
 239 then $\mathcal{D} = \mathcal{E}$.

240 Under our assumption that \mathcal{C} is essentially small, there is a *set* of t-
 241 structures on \mathcal{C} (because t-structures correspond to aisles, and the latter
 242 are uniquely specified by certain subsets of the set of isomorphism classes of
 243 objects). In contrast, [47] shows that t-structures on the derived category of
 244 all abelian groups (not necessarily finitely-generated) form a proper class.

245 **Definition 2.2.** Let $T(\mathcal{C})$ be the poset of bounded t-structures on \mathcal{C} , ordered
 246 by inclusion of the aisles. Abusing notation write $\mathcal{D} \subseteq \mathcal{E}$ to mean $\mathcal{D}^{\leq 0} \subseteq \mathcal{E}^{\leq 0}$.

247 There is a natural action of \mathbb{Z} on $T(\mathcal{C})$ given by shifting: we write $\mathcal{D}[n]$ for
 248 the t-structure $(\mathcal{D}^{\leq -n}, \mathcal{D}^{\geq -n+1})$. Note that $\mathcal{D}[1] \subseteq \mathcal{D}$, and not vice versa.

249 **2.3. Torsion structures and tilting.** The notion of torsion structure, also
 250 known as a torsion/torsion-free pair, is an abelian analogue of that of t-
 251 structure; the notions are related by the process of tilting.

252 **Definition 2.3.** A *torsion structure* on an abelian category \mathcal{A} is an ordered
 253 pair $\mathcal{T} = (\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 1})$ of subcategories satisfying

- 254 (1) $\text{Hom}_{\mathcal{A}}(t, t') = 0$ whenever $t \in \mathcal{T}^{\leq 0}$ and $t' \in \mathcal{T}^{\geq 1}$;
- 255 (2) for any $a \in \mathcal{A}$ there is a short exact sequence $0 \rightarrow t \rightarrow a \rightarrow t' \rightarrow 0$
 256 with $t \in \mathcal{T}^{\leq 0}$ and $t' \in \mathcal{T}^{\geq 1}$.

257 The subcategory $\mathcal{T}^{\leq 0}$ is given by the *torsion theory* of \mathcal{T} , and $\mathcal{T}^{\geq 1}$ by the
 258 *torsion-free theory*; the motivating example is the subcategories of torsion
 259 and torsion-free abelian groups.

260 The short exact sequence $0 \rightarrow t \rightarrow a \rightarrow t' \rightarrow 0$ is unique up to isomor-
 261 phism. The first term determines a right adjoint to the inclusion $\mathcal{T}^{\leq 0} \hookrightarrow \mathcal{A}$
 262 and the last term a left adjoint to the inclusion $\mathcal{T}^{\geq 1} \hookrightarrow \mathcal{A}$. It follows that
 263 $\mathcal{T}^{\leq 0}$ is closed under factors, extensions and coproducts and that $\mathcal{T}^{\geq 1}$ is
 264 closed under subobjects, extensions and products. Torsion structures in \mathcal{A} ,
 265 ordered by inclusion of their torsion theories, form a poset. It is bounded,
 266 with minimal element $(0, \mathcal{A})$ and maximal element $(\mathcal{A}, 0)$.

267 **Proposition 2.4** ([25, Proposition 2.1], [7, Theorem 3.1]). *Let \mathcal{D} be a t-*
 268 *structure on a triangulated category \mathcal{C} . Then there is a canonical isomor-*
 269 *phism between the poset of torsion structures in the heart \mathcal{D}^0 and the interval*
 270 *$[\mathcal{D}, \mathcal{D}[-1]]_{\subseteq}$ in $\text{T}(\mathcal{C})$ consisting of t-structures \mathcal{E} with $\mathcal{D} \subseteq \mathcal{E} \subseteq \mathcal{D}[-1]$.*

271 Let \mathcal{D} be a t-structure on a triangulated category \mathcal{C} . It follows from
 272 Proposition 2.4 that a torsion structure \mathcal{T} in the heart \mathcal{D}^0 determines a new
 273 t-structure

$$L_{\mathcal{T}}\mathcal{D} = (\langle \mathcal{D}^{\leq 0}, \mathcal{T}^{\leq 1} \rangle, \langle \mathcal{T}^{\geq 2}, \mathcal{D}^{\geq 2} \rangle)$$

274 called the *left tilt* of \mathcal{D} at \mathcal{T} , where by definition $\mathcal{T}^{\leq k} = \mathcal{T}^{\leq 0}[-k]$ and
 275 similarly $\mathcal{T}^{\geq k} = \mathcal{T}^{\geq 1}[1-k]$. The heart of the left tilt is $\langle \mathcal{T}^{\leq 1}, \mathcal{T}^{\geq 1} \rangle$ and
 276 $\mathcal{D} \subseteq L_{\mathcal{T}}\mathcal{D} \subseteq \mathcal{D}[-1]$. The shifted t-structure $R_{\mathcal{T}}\mathcal{D} = L_{\mathcal{T}}\mathcal{D}[1]$ is called the
 277 *right tilt* of \mathcal{D} at \mathcal{T} . It has heart $\langle \mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0} \rangle$ and $\mathcal{D}[1] \subseteq R_{\mathcal{T}}\mathcal{D} \subseteq \mathcal{D}$. Left and
 278 right tilting are inverse to one another: $(\mathcal{T}^{\geq 1}, \mathcal{T}^{\leq 1})$ is a torsion structure
 279 on the heart of $L_{\mathcal{T}}\mathcal{D}$, and right tilting with respect to this we recover the
 280 original t-structure. Similarly, $(\mathcal{T}^{\geq 0}, \mathcal{T}^{\leq 0})$ is a torsion structure on the heart
 281 of $R_{\mathcal{T}}\mathcal{D}$, and left tilting with respect to this we return to \mathcal{D} . Since there
 282 is a correspondence between bounded t-structures and their hearts we will,
 283 where convenient, speak of the left or right tilt of a heart.

284 **Definition 2.5.** Let the *tilting poset* $\text{Tilt}(\mathcal{C})$ be the poset of t-structures
 285 with $\mathcal{D} \leq \mathcal{E}$ if and only if there is a finite sequence of left tilts from \mathcal{D} to \mathcal{E} .

286 **Remark 2.6.** An easy induction shows that if $\mathcal{D} \leq \mathcal{E}$ then $\mathcal{D} \subseteq \mathcal{E} \subseteq \mathcal{D}[-k]$
 287 for some $k \in \mathbb{N}$.

288 It follows that the identity on elements is a map of posets $\text{Tilt}(\mathcal{C}) \rightarrow \text{T}(\mathcal{C})$.
 289 By Proposition 2.4, if $\mathcal{D} \subseteq \mathcal{E} \subseteq \mathcal{D}[-1]$ then $\mathcal{D} \leq \mathcal{E} \iff \mathcal{D} \subseteq \mathcal{E}$, so that the
 290 map induces an isomorphism $[\mathcal{D}, \mathcal{D}[-1]]_{\subseteq} \cong [\mathcal{D}, \mathcal{D}[-1]]_{\subseteq}$.

Lemma 2.7. *Suppose \mathcal{D} and \mathcal{E} are in the same component of $\text{Tilt}(\mathcal{C})$. Then $\mathcal{F} \leq \mathcal{D}, \mathcal{E} \leq \mathcal{G}$ for some \mathcal{F}, \mathcal{G} . (We do not claim that \mathcal{F} and \mathcal{G} are the infimum and supremum, simply that lower and upper bounds exist.)*

Proof. If \mathcal{D} and \mathcal{E} are left tilts of some t-structure \mathcal{H} then they are right tilts of $\mathcal{H}[-1]$, and vice versa. It follows that we can replace an arbitrary sequence of left and right tilts connecting \mathcal{D} with \mathcal{E} by a sequence of left tilts followed by a sequence of right tilts, or vice versa. \square

2.4. Algebraic t-structures. We say an abelian category is *algebraic* if it is a length category with finitely many isomorphism classes of simple objects. To spell this out, this means it is both Artinian and Noetherian so that every object has a finite composition series. By the Jordan-Hölder theorem, the graded object associated to such a composition series is unique up to isomorphism. For instance, the module category $\text{mod } A$ of a finite-dimensional algebra A is algebraic.

The classes of the simple objects in an algebraic abelian category form a basis for the Grothendieck group, which is isomorphic to \mathbb{Z}^n , where n is the number of such classes. A t-structure \mathcal{D} is *algebraic* if its heart \mathcal{D}^0 is. If \mathcal{C} admits an algebraic t-structure then the heart of any other t-structure on \mathcal{C} which is a length category must also have exactly n isomorphism classes of simple objects, and therefore must be algebraic, since the two hearts have isomorphic Grothendieck groups.

Let the *algebraic tilting poset* $\text{Tilt}_{\text{alg}}(\mathcal{C})$ be the poset consisting of the algebraic t-structures, with $\mathcal{D} \preceq \mathcal{E}$ when \mathcal{E} is obtained from \mathcal{D} by a finite sequence of left tilts, via algebraic t-structures. Clearly

$$\mathcal{D} \preceq \mathcal{E} \Rightarrow \mathcal{D} \leq \mathcal{E} \Rightarrow \mathcal{D} \subseteq \mathcal{E},$$

and there is an injective map of posets $\text{Tilt}_{\text{alg}}(\mathcal{C}) \rightarrow \text{Tilt}(\mathcal{C})$.

Remark 2.8. There is an alternative algebraic description of $\text{Tilt}_{\text{alg}}(\mathcal{C})$ when $\mathcal{C} = \mathcal{D}(A)$ is the bounded derived category of a finite-dimensional algebra A , of finite global dimension, over an algebraically-closed field. By [19, Lemma 4.1] the poset $\mathbb{P}_1(\mathcal{C})$ of silting subcategories in \mathcal{C} is the sub-poset of $\text{T}(\mathcal{C})^{\text{op}}$ consisting of the algebraic t-structures, and under this identification silting mutation in $\mathbb{P}_1(\mathcal{C})$ corresponds to (admissible) tilting in $\text{T}(\mathcal{C})^{\text{op}}$. Moreover, it follows from [2, §2.6] that the partial order in $\mathbb{P}_1(\mathcal{C})$ is generated by silting mutation, so that $\mathcal{D} \subseteq \mathcal{E} \iff \mathcal{D} \preceq \mathcal{E}$ for algebraic \mathcal{D} and \mathcal{E} . Hence $\text{Tilt}_{\text{alg}}(\mathcal{C}) \cong \mathbb{P}_1(\mathcal{C})^{\text{op}}$.

If A does not have finite global dimension, then a similar result holds but we must replace the poset of silting subcategories in \mathcal{C} , with the analogous poset in the bounded homotopy category of finitely-generated projective modules.

Lemma 2.9. *Suppose \mathcal{D} and \mathcal{E} are t-structures and that \mathcal{E} is algebraic. Then $\mathcal{E} \subseteq \mathcal{D}[-d]$ for some $d \in \mathbb{N}$.*

Proof. Since \mathcal{D} is bounded each simple object s of the heart \mathcal{E}^0 is in $\mathcal{D}^{\leq k_s}$ for some $k_s \in \mathbb{Z}$. Then $\mathcal{E}^0 \subseteq \mathcal{D}^{\leq d}$ for $d = \max_s \{k_s\}$ — the maximum exists since there are finitely many simple objects in \mathcal{E}^0 — and this implies $\mathcal{D} \subseteq \mathcal{D}[-d]$. \square

Remark 2.10. It follows that $BT(\mathcal{C})$ is contractible whenever \mathcal{C} admits an algebraic t-structure. To see this let $T_N(\mathcal{C})$ for $N \in \mathbb{N}$ be the sub-poset on $\{\mathcal{D} \mid \mathcal{E}[N] \subseteq \mathcal{D}\}$. Note that $BT_N(\mathcal{C})$ is the cone on the vertex corresponding to $\mathcal{E}[N]$, hence is contractible. The above lemma implies that $BT(\mathcal{C})$ is the colimit of the diagram

$$BT_0(\mathcal{C}) \hookrightarrow BT_1(\mathcal{C}) \hookrightarrow BT_2(\mathcal{C}) \hookrightarrow \dots$$

of contractible spaces. Hence it is also contractible.

Lemma 2.11. *Suppose \mathcal{D} and \mathcal{E} are in the same component of $\text{Tilt}_{\text{alg}}(\mathcal{C})$. Then $\mathcal{F} \preceq \mathcal{D}, \mathcal{E} \preceq \mathcal{G}$ for some \mathcal{F}, \mathcal{G} in that component.*

Proof. This is proved in exactly the same way as Lemma 2.7; note that all t-structures encountered in the construction will be algebraic. \square

It is not clear that the poset $T(\mathcal{C})$ of t-structures is always a lattice — see [10] for an example in which the naive meet (i.e. intersection) of t-structures is not itself a t-structure, and also [17] — and we do not claim that the lower and upper bounds of the previous lemma are infima or suprema. We do however have the following weaker result.

Lemma 2.12. *Suppose \mathcal{D} is algebraic (in fact it suffices for its heart to be a length category). Then for each $\mathcal{D} \subseteq \mathcal{E}, \mathcal{F} \subseteq \mathcal{D}[-1]$ there is a supremum $\mathcal{E} \vee \mathcal{F}$ and an infimum $\mathcal{E} \wedge \mathcal{F}$ in $T(\mathcal{C})$.*

Proof. We construct only the supremum $\mathcal{E} \vee \mathcal{F}$, the infimum is constructed similarly. We claim that $\langle \mathcal{E}^{\leq 0}, \mathcal{F}^{\leq 0} \rangle$ is the aisle of a bounded t-structure; it is clear that this t-structure must then be the supremum in $T(\mathcal{C})$.

Since $\mathcal{D} \subseteq \mathcal{E}, \mathcal{F} \subseteq \mathcal{D}[-1]$ we may work with the corresponding torsion structures $\mathcal{T}_{\mathcal{E}}$ and $\mathcal{T}_{\mathcal{F}}$ on \mathcal{D}^0 , and show that $\mathcal{T}^{\leq 0} = \langle \mathcal{T}_{\mathcal{E}}^{\leq 0}, \mathcal{T}_{\mathcal{F}}^{\leq 0} \rangle$ is a torsion theory, with associated torsion-free theory $\mathcal{T}^{\geq 1} = \mathcal{T}_{\mathcal{E}}^{\geq 1} \cap \mathcal{T}_{\mathcal{F}}^{\geq 1}$. Certainly $\text{Hom}_{\mathcal{C}}(t, t') = 0$ whenever $t \in \mathcal{T}^{\leq 0}$ and $t' \in \mathcal{T}^{\geq 1}$, so it suffices to show that any $d \in \mathcal{D}^0$ sits in a short exact sequence $0 \rightarrow t \rightarrow d \rightarrow t' \rightarrow 0$ with $t \in \mathcal{T}^{\leq 0}$ and $t' \in \mathcal{T}^{\geq 1}$. We do this in stages, beginning with the short exact sequence

$$0 \rightarrow e_0 \rightarrow d \rightarrow e'_0 \rightarrow 0$$

with $e_0 \in \mathcal{T}_{\mathcal{E}}^{\leq 0}$ and $e'_0 \in \mathcal{T}_{\mathcal{E}}^{\geq 1}$. Combining this with the short exact sequence $0 \rightarrow f_0 \rightarrow e'_0 \rightarrow f'_0 \rightarrow 0$ with $f_0 \in \mathcal{T}_{\mathcal{F}}^{\leq 0}$ and $f'_0 \in \mathcal{T}_{\mathcal{F}}^{\geq 1}$ we obtain a second short exact sequence

$$0 \rightarrow t \rightarrow d \rightarrow f'_0 \rightarrow 0$$

where t is an extension of e_0 and f_0 , and hence is in $\mathcal{T}^{\leq 0}$. Repeat this process, at each stage using the expression of the third term as an extension via alternately the torsion structures $\mathcal{T}_{\mathcal{E}}$ and $\mathcal{T}_{\mathcal{F}}$. This yields successive short exact sequences, each with middle term d and first term in $\mathcal{T}^{\leq 0}$, and such that the third term is a quotient of the third term of the previous sequence. Since \mathcal{D}^0 is a length category this process must stabilise. It does so when the third term has no subobject in either $\mathcal{T}_{\mathcal{E}}^{\leq 0}$ or $\mathcal{T}_{\mathcal{F}}^{\leq 0}$, i.e. when the third term is in $\mathcal{T}_{\mathcal{E}}^{\geq 1} \cap \mathcal{T}_{\mathcal{F}}^{\geq 1} = \mathcal{T}^{\geq 1}$. This exhibits the required short exact sequence and completes the proof. \square

In general, this cannot be used inductively to show that the components of $\text{Tilt}_{\text{alg}}(\mathcal{C})$ are lattices, since $\mathcal{E} \wedge \mathcal{F}$ and $\mathcal{E} \vee \mathcal{F}$ might not be algebraic. For the remainder of this section we impose an assumption that guarantees that they are: let $\text{Tilt}^\circ(\mathcal{C}) = \text{Tilt}_{\text{alg}}^\circ(\mathcal{C})$ be a component of the tilting poset consisting entirely of algebraic t-structures, equivalently a component of $\text{Tilt}_{\text{alg}}(\mathcal{C})$ closed under all tilts.

Lemma 2.13. *The component $\text{Tilt}^\circ(\mathcal{C})$ is a lattice. Infima and suprema in $\text{Tilt}^\circ(\mathcal{C})$ are also infima and suprema in $\text{T}(\mathcal{C})$.*

Proof. Suppose $\mathcal{E}, \mathcal{F} \in \text{Tilt}^\circ(\mathcal{C})$. As in Lemma 2.7 we can replace an arbitrary sequence of left and right tilts connecting \mathcal{E} with \mathcal{F} by one consisting of a sequence of left tilts followed by a sequence of right tilts, or vice versa, but now using the infima and suprema of Lemma 2.12 at each stage of the process. We can do this since $\text{Tilt}^\circ(\mathcal{C})$ consists entirely of algebraic t-structures, and therefore these infima and suprema are algebraic. Thus \mathcal{E} and \mathcal{F} have upper and lower bounds in $\text{Tilt}^\circ(\mathcal{C})$.

We now construct the infimum and supremum. First, convert the sequence of tilts from \mathcal{E} to \mathcal{F} into one of right followed by left tilts by the above process. Then if $\mathcal{E}, \mathcal{F} \subseteq \mathcal{G}$ the same is true for each t-structure along the new sequence. Now convert this new sequence to one of left tilts followed by right tilts, again by the above process. Inductively applying Lemma 2.12 shows that each t-structure in the resulting sequence is still bounded above in $\text{T}(\mathcal{C})$ by \mathcal{G} . In particular the t-structure \mathcal{H} reached after the final left tilt, and before the first right tilt, satisfies $\mathcal{E}, \mathcal{F} \preceq \mathcal{H} \subseteq \mathcal{G}$. It follows that $\mathcal{H} \in \text{Tilt}^\circ(\mathcal{C})$ is the supremum $\mathcal{E} \vee \mathcal{F}$ of \mathcal{E} and \mathcal{F} in $\text{T}(\mathcal{C})$.

To complete the proof we need to show that $\mathcal{E} \vee \mathcal{F} \preceq \mathcal{G}$ whenever \mathcal{G} is in $\text{Tilt}^\circ(\mathcal{C})$ and $\mathcal{E}, \mathcal{F} \preceq \mathcal{G}$. This follows since $\mathcal{E} \vee \mathcal{F} \preceq (\mathcal{E} \vee \mathcal{F}) \vee \mathcal{G} = \mathcal{G}$.

The argument for the infimum is similar. \square

Lemma 2.14. *The following are equivalent:*

- (1) *Intervals of the form $[\mathcal{D}, \mathcal{D}[-1]]_{\preceq}$ in $\text{Tilt}^\circ(\mathcal{C})$ are finite.*
- (2) *All closed bounded intervals in $\text{Tilt}^\circ(\mathcal{C})$ are finite.*

Proof. Assume that intervals of the form $[\mathcal{D}, \mathcal{D}[-1]]_{\preceq}$ in $\text{Tilt}_{\text{alg}}(\mathcal{C})$ are finite. Given $\mathcal{D} \preceq \mathcal{E}$ in $\text{Tilt}^\circ(\mathcal{C})$ recall that $\mathcal{E} \subseteq \mathcal{D}[-d]$ for some $d \in \mathbb{N}$ by Lemma 2.9, so that

$$\mathcal{D} \preceq \mathcal{E} \preceq \mathcal{E} \vee \mathcal{D}[-d] = \mathcal{D}[-d].$$

Hence it suffices to show that intervals of the form $[\mathcal{D}, \mathcal{D}[-d]]_{\preceq}$ are finite. We prove this by induction on d . The case $d = 1$ is true by assumption. Suppose it is true for $d < k$. In diagrams it will be convenient to use the notation $\mathcal{E} \rightsquigarrow \mathcal{F}$ to mean \mathcal{F} is a left tilt of \mathcal{E} .

By definition of $\text{Tilt}_{\text{alg}}(\mathcal{C})$ any element of the interval $[\mathcal{D}, \mathcal{D}[-k]]_{\preceq}$ sits in a chain of tilts $\mathcal{D} = \mathcal{D}_0 \rightsquigarrow \mathcal{D}_1 \rightsquigarrow \dots \rightsquigarrow \mathcal{D}_r = \mathcal{D}[-k]$ via algebraic t-structures. This can be extended to a diagram

$$\begin{array}{ccccccc}
 \mathcal{D} = \mathcal{D}_0 & \rightsquigarrow & \mathcal{D}_1 & \rightsquigarrow & \mathcal{D}_2 & \rightsquigarrow & \dots & \rightsquigarrow & \mathcal{D}_{r-1} & \rightsquigarrow & \mathcal{D}_r = \mathcal{D}[-k] \\
 & \searrow & \downarrow & & \downarrow & & & & \downarrow & & \nearrow \\
 & & \mathcal{D}'_1 & \rightsquigarrow & \mathcal{D}'_2 & \rightsquigarrow & \dots & \rightsquigarrow & \mathcal{D}'_{r-1} & &
 \end{array}$$

415 of algebraic t-structures and tilts, where $\mathcal{D}'_1 = \mathcal{D}[-1]$, so that $\mathcal{D}_1 \rightsquigarrow \mathcal{D}'_1$ as
 416 shown, and $\mathcal{D}'_i = \mathcal{D}_i \vee \mathcal{D}'_{i-1}$ is constructed inductively. The only point that
 417 requires elaboration is the existence of the tilt $\mathcal{D}'_{r-1} \rightsquigarrow \mathcal{D}_r$. First note that
 418 $\mathcal{D}'_1, \mathcal{D}_2 \preceq \mathcal{D}_r$ so that $\mathcal{D}'_2 = \mathcal{D}_2 \vee \mathcal{D}'_1 \preceq \mathcal{D}_r$ too. By induction $\mathcal{D}'_{r-1} \preceq \mathcal{D}_r$.
 419 Since

$$\mathcal{D}_r[1] \preceq \mathcal{D}_{r-1} \preceq \mathcal{D}'_{r-1} \preceq \mathcal{D}_r$$

420 \mathcal{D}_r is a left tilt of \mathcal{D}'_{r-1} by Proposition 2.4.

421 The existence of the above diagram shows that each element of the interval
 422 $[\mathcal{D}, \mathcal{D}[-k]]_{\preceq}$ is a right tilt of some element of the interval $[\mathcal{D}[-1], \mathcal{D}[-k]]_{\preceq}$.
 423 By induction the latter has only finitely many elements, and by assumption
 424 each of these has only finitely many right tilts. This establishes the first
 425 implication. The converse is obvious. \square

426 **2.5. Simple tilts.** Suppose \mathcal{D} is an algebraic t-structure. Then each simple
 427 object $s \in \mathcal{D}^0$ determines two torsion structures on the heart, namely
 428 $(\langle s \rangle, \langle s \rangle^\perp)$ and $({}^\perp \langle s \rangle, \langle s \rangle)$. These are respectively minimal and maximal non-
 429 trivial torsion structures in \mathcal{D}^0 . We say the left tilt at the former, and the
 430 right tilt at the latter, are *simple*. We use the abbreviated notation $L_s \mathcal{D}$
 431 and $R_s \mathcal{D}$ respectively for these tilts.

432 More generally we have the following notions. A torsion structure \mathcal{T} is
 433 *hereditary* if $t \in \mathcal{T}^{\leq 0}$ implies all subobjects of t are in $\mathcal{T}^{\leq 0}$. It is *co-hereditary*
 434 if $t \in \mathcal{T}^{\geq 1}$ implies all quotients of t are in $\mathcal{T}^{\geq 1}$. It follows that the aisle of a
 435 hereditary torsion, dually the coaisle of a cohereditary torsion structure, are
 436 Serre subcategories. When \mathcal{T} is a torsion structure on an algebraic abelian
 437 category then the hereditary torsion structures are those of the form (S, S^\perp)
 438 where the torsion theory $S = \langle s_1, \dots, s_k \rangle$ is generated by a subset of the
 439 simple objects. Dually, the co-hereditary torsion structures are those of
 440 the form $({}^\perp S, S)$. We use the abbreviated notation $L_S \mathcal{D}$ for the left tilt at
 441 (S, S^\perp) and $R_S \mathcal{D}$ for the right tilt at $({}^\perp S, S)$. Note that, in the notation of
 442 the previous section, $L_S \mathcal{D} \wedge L_{S'} \mathcal{D} = L_{S \cap S'} \mathcal{D}$ and $L_S \mathcal{D} \vee L_{S'} \mathcal{D} = L_{S \cup S'} \mathcal{D}$.

443 In general a tilt, even a simple tilt, of an algebraic t-structure need not
 444 be algebraic. However, if the heart is *rigid*, i.e. the simple objects have no
 445 self-extensions, then [33, Proposition 5.4] shows that the tilted t-structure
 446 is also algebraic. We will see later in Lemma 4.2 that the same holds if the
 447 heart has only finitely many isomorphism classes of indecomposable objects.

448 **2.6. Stability conditions.** Let \mathcal{C} be a triangulated category and $K\mathcal{C}$ be its
 449 Grothendieck group. A *stability condition* (Z, \mathcal{P}) on \mathcal{C} [12, Definition 1.1]
 450 consists of a group homomorphism $Z: K\mathcal{C} \rightarrow \mathbb{C}$ and full additive subcate-
 451 gories $\mathcal{P}(\varphi)$ of \mathcal{C} for each $\varphi \in \mathbb{R}$ satisfying

- 452 (1) if $c \in \mathcal{P}(\varphi)$ then $Z(c) = m(c) \exp(i\pi\varphi)$ where $m(c) \in \mathbb{R}_{>0}$;
- 453 (2) $\mathcal{P}(\varphi + 1) = \mathcal{P}(\varphi)[1]$ for each $\varphi \in \mathbb{R}$;
- 454 (3) if $c \in \mathcal{P}(\varphi)$ and $c' \in \mathcal{P}(\varphi')$ with $\varphi > \varphi'$ then $\text{Hom}(c, c') = 0$;
- 455 (4) for each nonzero object $c \in \mathcal{C}$ there is a finite collection of triangles

$$\begin{array}{ccccccc}
 0 = c_0 & \longrightarrow & c_1 & \longrightarrow & \cdots & \longrightarrow & c_{n-1} & \longrightarrow & c_n = c \\
 & & \searrow & & & & \nwarrow & & \downarrow \\
 & & & & & & & & b_n \\
 & & & & & & & & \uparrow \\
 & & & & & & & & b_1
 \end{array}$$

457 with $b_j \in \mathcal{P}(\varphi_j)$ where $\varphi_1 > \cdots > \varphi_n$.

458 The homomorphism Z is known as the *central charge* and the objects of
 459 $\mathcal{P}(\varphi)$ are said to be *semistable of phase* φ . The objects b_j are known as the
 460 *semistable factors* of c . We define $\varphi^+(c) = \varphi_1$ and $\varphi^-(c) = \varphi_n$. The *mass*
 461 of c is defined to be $m(c) = \sum_{i=1}^n m(b_i)$.

462 For an interval $(a, b) \subseteq \mathbb{R}$ we set $\mathcal{P}(a, b) = \langle c \in \mathcal{C} : \varphi(c) \in (a, b) \rangle$, and
 463 similarly for half-open or closed intervals. Each stability condition σ has an
 464 associated bounded t-structure $\mathcal{D}_\sigma = (\mathcal{P}(0, \infty), \mathcal{P}(-\infty, 0])$ with heart $\mathcal{D}_\sigma^0 =$
 465 $\mathcal{P}(0, 1]$. Conversely, if we are given a bounded t-structure on \mathcal{C} together with
 466 a stability function on the heart with the Harder–Narasimhan property —
 467 the abelian analogue of property (4) above — then this determines a stability
 468 condition on \mathcal{C} [12, Proposition 5.3].

A stability condition is *locally-finite* if we can find $\epsilon > 0$ such that the
 quasi-abelian category $\mathcal{P}(t - \epsilon, t + \epsilon)$, generated by semistable objects with
 phases in $(t - \epsilon, t + \epsilon)$, has finite length (see [12, Definition 5.7]). The
 set of locally-finite stability conditions can be topologised so that it is a,
 possibly infinite-dimensional, complex manifold, which we denote $\text{Stab}(\mathcal{C})$
 [12, Theorem 1.2]. The topology arises from the (generalised) metric

$$d(\sigma, \tau) = \sup_{0 \neq c \in \mathcal{C}} \max \left(|\varphi_\sigma^-(c) - \varphi_\tau^-(c)|, |\varphi_\sigma^+(c) - \varphi_\tau^+(c)|, \left| \log \frac{m_\sigma(c)}{m_\tau(c)} \right| \right)$$

469 which takes values in $[0, \infty]$. It follows that for fixed $0 \neq c \in \mathcal{C}$ the mass
 470 $m_\sigma(c)$, and lower and upper phases $\varphi_\sigma^-(c)$ and $\varphi_\sigma^+(c)$ are continuous functions
 471 $\text{Stab}(\mathcal{C}) \rightarrow \mathbb{R}$. The projection

$$\pi: \text{Stab}(\mathcal{C}) \rightarrow \text{Hom}(K\mathcal{C}, \mathbb{C}) : (Z, \mathcal{P}) \mapsto Z$$

472 is a local homeomorphism.

The group $\text{Aut}(\mathcal{C})$ of auto-equivalences acts continuously on the space
 $\text{Stab}(\mathcal{C})$ of stability conditions with an automorphism α acting by

$$(Z, \mathcal{P}) \mapsto (Z \circ \alpha^{-1}, \alpha(\mathcal{P})). \quad (1)$$

473 There is also a smooth right action of the universal cover G of $GL_2^+ \mathbb{R}$. An
 474 element $g \in G$ corresponds to a pair (T_g, θ_g) where T_g is the projection of
 475 g to $GL_2^+ \mathbb{R}$ under the covering map and $\theta_g: \mathbb{R} \rightarrow \mathbb{R}$ is an increasing map
 476 with $\theta_g(t+1) = \theta_g(t) + 1$ which induces the same map as T_g on the circle
 477 $\mathbb{R}/2\mathbb{Z} = \mathbb{R}^2 - \{0\}/\mathbb{R}_{>0}$. The action is given by

$$(Z, \mathcal{P}) \mapsto (T_g^{-1} \circ Z, \mathcal{P} \circ \theta_g). \quad (2)$$

478 (Here we think of the central charge as valued in \mathbb{R}^2 .) This action preserves
 479 the semistable objects, and also preserves the Harder–Narasimhan filtra-
 480 tions of all objects. The subgroup consisting of pairs with T conformal is
 481 isomorphic to \mathbb{C} with $\lambda \in \mathbb{C}$ acting via

$$(Z, \mathcal{P}) \mapsto (\exp(-i\pi\lambda)Z, \mathcal{P}(\varphi + \text{Re } \lambda))$$

482 i.e. by rotating the phases and rescaling the masses of semistable objects.
 483 This action is free and preserves the metric. The action of $1 \in \mathbb{C}$ corresponds
 484 to the action of the shift automorphism [1].

485 **Lemma 2.15.** *For any $g \in G$ the t-structures $\mathcal{D}_{g \cdot \sigma}$ and \mathcal{D}_σ are related by a*
 486 *finite sequence of tilts.*

487 *Proof.* This is simple to verify directly by considering the way in which G
 488 acts on phases. Alternatively, note that G is connected, so that σ and $g \cdot \sigma$
 489 are in the same component of $\text{Stab}(\mathcal{C})$. Hence by [53, Corollary 5.2] the
 490 t-structures \mathcal{D}_σ and \mathcal{D}_τ are related by a finite sequence of tilts. \square

491 **2.7. Cellular stratified spaces.** A CW-cellular stratified space, in the
 492 sense of [23], is a generalisation of a CW-complex in which non-compact
 493 cells are permitted. In §3 we will show that (parts of) stability spaces have
 494 this structure, and use it to show their contractibility. Here, we recall the
 495 definitions and results we will require.

496 A k -cell structure on a subspace e of a topological space X is a continuous
 497 map $\alpha: D \rightarrow X$ where $\text{int}(\mathbb{D}^k) \subseteq D \subseteq \mathbb{D}^k$ is a subset of the k -dimensional
 498 disk $\mathbb{D}^k \subset \mathbb{R}^k$ containing the interior, such that $\alpha(D) = \bar{e}$, the restriction of
 499 α to $\text{int}(\mathbb{D}^k)$ is a homeomorphism onto e , and α does not extend to a map
 500 with these properties defined on any larger subset of \mathbb{D}^k . We refer to e as a
 501 cell and to α as a *characteristic map* for e .

502 **Definition 2.16.** A *cellular stratification* of a topological space X consists
 503 of a filtration

$$\emptyset = X_{-1} \subseteq X_0 \subseteq X_1 \subseteq \cdots \subseteq X_k \subseteq \cdots$$

504 by subspaces, with $X = \bigcup_{k \in \mathbb{N}} X_k$, such that $X_k - X_{k-1} = \bigsqcup_{\lambda \in \Lambda_k} e_\lambda$ is a
 505 disjoint union of k -cells for each $k \in \mathbb{N}$. A *CW-cellular stratification* is a
 506 cellular stratification satisfying the further conditions that

- 507 (1) the stratification is closure-finite, i.e. the boundary $\partial e = \bar{e} - e$ of
 508 any k -cell is contained in a union of finitely many lower-dimensional
 509 cells;
- 510 (2) X has the weak topology determined by the closures \bar{e} of the cells
 511 in the stratification, i.e. a subset A of X is closed if, and only if, its
 512 intersection with each \bar{e} is closed.

513 When the domain of each characteristic map is the entire disk then a
 514 CW-cellular stratification is nothing but a CW-complex structure on X .
 515 Although the collection of cells and characteristic maps is part of the data
 516 of a cellular stratified space we will suppress it from our notation for ease-of-
 517 reading. Since we never consider more than one stratification of any given
 518 topological space there is no possibility for confusion.

519 A cellular stratification is said to be *regular* if each characteristic map is a
 520 homeomorphism, and *normal* if the boundary of each cell is a union of lower-
 521 dimensional cells. A regular, normal cellular stratification induces cellular
 522 stratifications on the domain of the characteristic map of each of its cells.
 523 Finally, we say a CW-cellular stratification is *regular and totally-normal* if
 524 it is regular, normal, and in addition for each cell e_λ with characteristic
 525 map $\alpha_\lambda: D_\lambda \rightarrow X$ the induced cellular stratification of $\partial D_\lambda = D_\lambda - \text{int}(\mathbb{D}^k)$
 526 extends to a regular CW-complex structure on $\partial \mathbb{D}^k$. (The definition of
 527 totally-normal CW-cellular stratification in [23] is more subtle, as it handles
 528 the non-regular case too, but it reduces to the above for regular stratifica-
 529 tions. A regular CW-complex is totally-normal, but regularity alone does
 530 not even entail normality for a CW-cellular stratified space.) Any union
 531 of strata in a regular, totally-normal CW-cellular stratified space is itself a
 532 regular, totally-normal CW-cellular stratified space.

533 A normal cellular stratified space X has a *poset of strata* (or face poset)
 534 $P(X)$ whose elements are the cells, and where $e_\lambda \leq e_\mu \iff e_\lambda \subseteq \overline{e_\mu}$. When
 535 X is a regular CW-complex there is a homeomorphism from the classifying
 536 space $BP(X)$ to X . More generally,

537 **Theorem 2.17** ([23, Theorem 2.50]). *Suppose X is a regular, totally-normal*
 538 *CW-cellular stratified space. Then $BP(X)$ embeds in X as a strong deformation*
 539 *retract, in particular there is a homotopy equivalence $X \simeq BP(X)$.*

540

3. ALGEBRAIC STABILITY CONDITIONS

541 We say a stability condition σ is *algebraic* if the corresponding t-structure
 542 \mathcal{D}_σ is algebraic. Let $\text{Stab}_{\text{alg}}(\mathcal{C}) \subseteq \text{Stab}(\mathcal{C})$ be the subspace of algebraic
 543 stability conditions.

544 Write $S_{\mathcal{D}} = \{\sigma \in \text{Stab}(\mathcal{C}) : \mathcal{D}_\sigma = \mathcal{D}\}$ for the set of stability conditions
 545 with associated t-structure \mathcal{D} . When \mathcal{D} is algebraic, a stability condition in
 546 $S_{\mathcal{D}}$ is uniquely determined by a choice of central charge in

$$\mathbb{H}_- = \{r \exp(i\pi\theta) \in \mathbb{C} : r > 0 \text{ and } \theta \in (0, 1]\} \quad (3)$$

547 for each simple object in the heart [14, Lemma 5.2]. Hence, in this case, an
 548 ordering of the simple objects determines an isomorphism $S_{\mathcal{D}} \cong (\mathbb{H}_-)^n$. In
 549 particular, if \mathcal{C} has an algebraic t-structure then $\text{Stab}_{\text{alg}}(\mathcal{C}) \neq \emptyset$.

550 The action of $\text{Aut}(\mathcal{C})$ on $\text{Stab}(\mathcal{C})$ restricts to an action on the subspace
 551 $\text{Stab}_{\text{alg}}(\mathcal{C})$. In contrast $\text{Stab}_{\text{alg}}(\mathcal{C})$ need not be preserved by the action of \mathbb{C}
 552 on $\text{Stab}(\mathcal{C})$. The action of $i\mathbb{R} \subseteq \mathbb{C}$ uniformly rescales the masses of semistable
 553 objects; this does not change the associated t-structure and so preserves
 554 $\text{Stab}_{\text{alg}}(\mathcal{C})$. However, $\mathbb{R} \subseteq \mathbb{C}$ acts by rotating the phases of semistables. Thus
 555 the action of $\lambda \in \mathbb{R}$ alters the t-structure by a finite sequence of tilts, and can
 556 result in a non-algebraic t-structure. In fact, the union of orbits $\mathbb{C} \cdot \text{Stab}_{\text{alg}}(\mathcal{C})$
 557 consists of those stability conditions σ for which $(\mathcal{P}_\sigma(\theta, \infty), \mathcal{P}_\sigma(-\infty, \theta])$ is an
 558 algebraic t-structure for some $\theta \in \mathbb{R}$. The choice of $\theta = 0$ for *the* associated
 559 t-structure is purely conventional. If we define

$$\text{Stab}_{\text{alg}}^\theta(\mathcal{C}) = \{\sigma \in \text{Stab}(\mathcal{C}) : (\mathcal{P}_\sigma(\theta, \infty), \mathcal{P}_\sigma(-\infty, \theta]) \text{ is algebraic}\}$$

560 then there is a commutative diagram

$$\begin{array}{ccc} \text{Stab}_{\text{alg}}(\mathcal{C}) & \hookrightarrow & \text{Stab}(\mathcal{C}) \\ \downarrow & & \downarrow \sigma \mapsto \theta \cdot \sigma \\ \text{Stab}_{\text{alg}}^\theta(\mathcal{C}) & \hookrightarrow & \text{Stab}(\mathcal{C}) \end{array}$$

562 in which the vertical maps are homeomorphisms. So $\text{Stab}_{\text{alg}}^\theta(\mathcal{C})$ is indepen-
 563 dent up to homeomorphism of the choice of $\theta \in \mathbb{R}$, but the way in which it
 564 is embedded in $\text{Stab}(\mathcal{C})$ is not.

565 **Lemma 3.1.** *Suppose $\text{Stab}_{\text{alg}}(\mathcal{C}) \neq \emptyset$. Then the space of algebraic stability*
 566 *conditions is contained in the union of full components of $\text{Stab}(\mathcal{C})$, i.e. those*
 567 *components locally homeomorphic to $\text{Hom}(K\mathcal{C}, \mathbb{C})$. A stability condition σ*
 568 *in a full component of $\text{Stab}(\mathcal{C})$ is algebraic if and only if $\mathcal{P}_\sigma(0, \epsilon) = \emptyset$ for*
 569 *some $\epsilon > 0$.*

570 *Proof.* The assumption that $\text{Stab}_{\text{alg}}(\mathcal{C}) \neq \emptyset$ implies that $K\mathcal{C} \cong \mathbb{Z}^n$ for some
 571 $n \in \mathbb{N}$. It follows from the description of $S_{\mathcal{D}}$ for algebraic \mathcal{D} above that any
 572 component containing an algebraic stability condition is full.

573 Suppose \mathcal{D} is algebraic. Then for any $\sigma \in S_{\mathcal{D}}$ the simple objects are
 574 semistable. Since there are finitely many simple objects there is one, s say,
 575 with minimal phase $\varphi_{\sigma}^{\pm}(s) = \epsilon > 0$. It follows that $\mathcal{P}_{\sigma}(0, \epsilon) = \emptyset$.

576 Conversely, suppose $\mathcal{P}_{\sigma}(0, \epsilon) = \emptyset$ for some stability condition σ in a full
 577 component. Then the heart $\mathcal{P}_{\sigma}(0, 1] = \mathcal{P}_{\sigma}(\epsilon, 1]$. Since $1 - \epsilon < 1$ we can
 578 apply [13, Lemma 4.5] to deduce that the heart of σ is an abelian length
 579 category. It follows that the heart has n simple objects (forming a basis of
 580 $K\mathcal{C}$), and hence is algebraic. \square

581 **Lemma 3.2.** *The interior of $S_{\mathcal{D}}$ is non-empty precisely when \mathcal{D} is algebraic.*

582 *Proof.* The explicit description of $S_{\mathcal{D}}$ for algebraic \mathcal{D} above shows that the
 583 interior is non-empty in this case. Conversely, suppose \mathcal{D} is not algebraic and
 584 $\sigma \in S_{\mathcal{D}}$. Then by Lemma 3.1 there are σ -semistable objects of arbitrarily
 585 small strictly positive phase. It follows that the \mathbb{C} -orbit through σ contains
 586 a sequence of stability conditions not in $S_{\mathcal{D}}$ with limit σ . Hence σ is not in
 587 the interior of $S_{\mathcal{D}}$. Since σ was arbitrary the latter must be empty. \square

588 **Corollary 3.3.** *The subset $\mathbb{C} \cdot \text{Stab}_{\text{alg}}(\mathcal{C}) \subseteq \text{Stab}(\mathcal{C})$ is open, and when non-*
 589 *empty consists of those stability conditions in full components of $\text{Stab}(\mathcal{C})$ for*
 590 *which the phases of semistable objects are not dense in \mathbb{R} .*

591 *Proof.* Suppose $\text{Stab}_{\text{alg}}(\mathcal{C}) \neq \emptyset$. Then $K\mathcal{C} \cong \mathbb{Z}^n$ for some n . A stability
 592 condition $\sigma \in \mathbb{C} \cdot \text{Stab}_{\text{alg}}(\mathcal{C})$ clearly lies in a component of $\text{Stab}(\mathcal{C})$ meeting
 593 $\text{Stab}_{\text{alg}}(\mathcal{C})$, and hence in a full component. By Lemma 3.1, if σ is in a full
 594 component then $\sigma \in \mathbb{C} \cdot \text{Stab}_{\text{alg}}(\mathcal{C})$ if and only if $\mathcal{P}_{\sigma}(t, t + \epsilon) = \emptyset$ for some
 595 $t \in \mathbb{R}$ and $\epsilon > 0$, equivalently if and only if the phases of semistable objects
 596 are not dense in \mathbb{R} .

597 To see that $\mathbb{C} \cdot \text{Stab}_{\text{alg}}(\mathcal{C})$ is open note that if $\sigma \in \mathbb{C} \cdot \text{Stab}_{\text{alg}}(\mathcal{C})$ and
 598 $d(\sigma, \tau) < \epsilon/4$ then $\mathcal{P}_{\sigma}(t + \epsilon/4, t + 3\epsilon/4) = \emptyset$ and so $\tau \in \mathbb{C} \cdot \text{Stab}_{\text{alg}}(\mathcal{C})$ too. \square

599 **Example 3.4.** Let X be a smooth complex projective algebraic curve with
 600 genus $g(X) > 0$. Then the space $\text{Stab}(X)$ of stability conditions on the
 601 bounded derived category of coherent sheaves on X is a single orbit of the G -
 602 action (2) through the stability condition with associated heart the coherent
 603 sheaves, and central charge $Z(\mathcal{E}) = -\deg \mathcal{E} + i \text{rank } \mathcal{E}$ — see [12, Theorem
 604 9.1] for $g(X) = 1$ and [37, Theorem 2.7] for $g(X) > 1$. It follows from the
 605 fact that there are semistable sheaves of any rational slope when $g(X) > 0$
 606 that the phases of semistable objects are dense for every stability condition
 607 in $\text{Stab}(X)$. Hence $\text{Stab}_{\text{alg}}(\mathcal{D}(X)) = \emptyset$. In fact this is true quite generally,
 608 since for ‘most’ varieties the Grothendieck group $K(X) = K(\mathcal{D}(X)) \not\cong \mathbb{Z}^n$.

609 **Example 3.5.** Let Q be a finite connected quiver, and $\text{Stab}(Q)$ the space of
 610 stability conditions on the bounded derived category of its finite-dimensional
 611 representations over an algebraically-closed field. When Q has underlying
 612 graph of ADE Dynkin type, the phases of semistable objects form a discrete
 613 set [21, Lemma 3.13]; when it has extended ADE Dynkin type, the phases
 614 either form a discrete set or have accumulation points $t + \mathbb{Z}$ for some $t \in \mathbb{R}$
 615 (all cases occur) [21, Corollary 3.15]; for any other acyclic Q there exists a

family of stability conditions for which the phases are dense in some non-empty open interval [21, Proposition 3.32]; and for Q with oriented loops there exist stability conditions for which the phases of semistable objects are dense in \mathbb{R} by [21, Remark 3.33]. It follows that $\text{Stab}_{\text{alg}}(Q) = \text{Stab}(Q)$ only in the Dynkin case; that $\mathbb{C} \cdot \text{Stab}_{\text{alg}}(Q) = \text{Stab}(Q)$ in the Dynkin or extended Dynkin cases; and that $\mathbb{C} \cdot \text{Stab}_{\text{alg}}(Q) \neq \text{Stab}(Q)$ when Q has oriented loops. For a general acyclic quiver, we do not know whether $\mathbb{C} \cdot \text{Stab}_{\text{alg}}(Q) = \text{Stab}(Q)$ or not.

Remark 3.6. The density of the phases of semistable objects for a stability condition is an important consideration in other contexts too. [53, Proposition 4.1] states that if phases for σ are dense in \mathbb{R} then the orbit of the universal cover G of $GL_2^+(\mathbb{R})$ through σ is free, and the induced metric on the quotient $G \cdot \sigma / \mathbb{C} \cong G / \mathbb{C} \cong \mathbb{H}$ of the orbit is half the standard hyperbolic metric.

Lemma 3.7. *Suppose there exists a uniform lower bound on the maximal phase gap of algebraic stability conditions, i.e. that there exists $\delta > 0$ such that for each $\sigma \in \text{Stab}_{\text{alg}}(\mathcal{C})$ there exists $\varphi \in \mathbb{R}$ with $\mathcal{P}_\sigma(\varphi - \delta, \varphi + \delta) = \emptyset$. Then $\mathbb{C} \cdot \text{Stab}_{\text{alg}}(\mathcal{C})$ is closed, and hence is a union of components of $\text{Stab}(\mathcal{C})$.*

Proof. Suppose $\sigma \in \overline{\mathbb{C} \cdot \text{Stab}_{\text{alg}}(\mathcal{C})} - \mathbb{C} \cdot \text{Stab}_{\text{alg}}(\mathcal{C})$. Let $\sigma_n \rightarrow \sigma$ be a sequence in $\mathbb{C} \cdot \text{Stab}_{\text{alg}}(\mathcal{C})$ with limit σ . Write φ_n^\pm for $\varphi_{\sigma_n}^\pm$ and so on.

Fix $\epsilon > 0$. There exists $N \in \mathbb{N}$ such that $d(\sigma_n, \sigma) < \epsilon$ for $n \geq N$. By Corollary 3.3 the phases of semistable objects for σ are dense in \mathbb{R} . Thus, given $\varphi \in \mathbb{R}$, we can find θ with $|\theta - \varphi| < \epsilon$ such that $\mathcal{P}_\sigma(\theta) \neq \emptyset$. So by [53, §3] there exists $0 \neq c \in \mathcal{C}$ such that $\varphi_n^\pm(c) \rightarrow \theta$. Hence $c \in \mathcal{P}_N(\theta - \epsilon, \theta + \epsilon) \subseteq \mathcal{P}_N(\varphi - 2\epsilon, \varphi + 2\epsilon)$. In particular the latter is non-empty. Since φ is arbitrary we obtain a contradiction by choosing $\epsilon < \delta/2$. Hence $\mathbb{C} \cdot \text{Stab}_{\text{alg}}(\mathcal{C})$ is closed. \square

Example 3.8. Let $\text{Stab}(\mathbb{P}^1)$ be the space of stability conditions on the bounded derived category $\mathcal{D}(\mathbb{P}^1)$ of coherent sheaves on \mathbb{P}^1 . [38, Theorem 1.1] identifies $\text{Stab}(\mathbb{P}^1) \cong \mathbb{C}^2$. In particular there is a unique component, and it is full. The category $\mathcal{D}(\mathbb{P}^1)$ is equivalent to the bounded derived category $\mathcal{D}(\widetilde{A}_1)$ of finite-dimensional representations of the Kronecker quiver \widetilde{A}_1 . In particular, $\text{Stab}_{\text{alg}}(\mathbb{P}^1)$ is non-empty. The Kronecker quiver has extended ADE Dynkin type, so by Example 3.5 the phases of semistable objects for any $\sigma \in \text{Stab}(\mathbb{P}^1)$ are either discrete or accumulate at the points $t + \mathbb{Z}$ for some $t \in \mathbb{R}$. The subspace $\text{Stab}(\mathbb{P}^1) - \text{Stab}_{\text{alg}}(\mathbb{P}^1)$ consists of those stability conditions with phases accumulating at $\mathbb{Z} \subseteq \mathbb{R}$. Therefore $\mathbb{C} \cdot \text{Stab}_{\text{alg}}(\mathbb{P}^1) = \text{Stab}(\mathbb{P}^1)$ and $\text{Stab}_{\text{alg}}(\mathbb{P}^1)$ is not closed. Neither is it open [52, p20]: there are convergent sequences of stability conditions whose phases accumulate at \mathbb{Z} such that the phase of each semistable object in the limiting stability condition is actually in \mathbb{Z} .

An explicit analysis of the semistable objects for each stability condition, as in [38], reveals that there is no lower bound on the maximum phase gap of algebraic stability conditions, so that whilst this condition is sufficient to ensure $\mathbb{C} \cdot \text{Stab}_{\text{alg}}(\mathcal{C}) = \text{Stab}(\mathcal{C})$ it is not necessary.

3.1. The stratification of algebraic stability conditions. In this section we define and study a natural stratification of $\text{Stab}_{\text{alg}}(\mathcal{C})$ with contractible strata. Suppose \mathcal{D} is an algebraic t-structure on \mathcal{C} . Then $S_{\mathcal{D}} \cong (\mathbb{H}_-)^n$ where $n = \text{rank}(K\mathcal{C})$. For a subset I of the simple objects in the heart \mathcal{D}^0 of \mathcal{D} we define a subset of $\text{Stab}(\mathcal{C})$

$$\begin{aligned} S_{\mathcal{D},I} &= \{\sigma : \mathcal{D} = \mathcal{D}_{\sigma}, \varphi_{\sigma}(s) = 1 \text{ for simple } s \in \mathcal{D}^0 \iff s \in I\} \\ &= \{\sigma : \mathcal{D} = \mathcal{D}_{\sigma}, \mathcal{P}_{\sigma}(1) = \langle I \rangle\} \\ &= \{\sigma : \mathcal{D} = (\mathcal{P}_{\sigma}(0, \infty), \mathcal{P}_{\sigma}(-\infty, 0]), L_I \mathcal{D} = (\mathcal{P}_{\sigma}[0, \infty), \mathcal{P}_{\sigma}(-\infty, 0))\}. \end{aligned}$$

Clearly $S_{\mathcal{D}} = \bigcup_I S_{\mathcal{D},I}$ and there is a decomposition

$$\text{Stab}_{\text{alg}}(\mathcal{C}) = \bigcup_{\mathcal{D} \text{ alg}} S_{\mathcal{D}} = \bigcup_{\mathcal{D} \text{ alg}} \left(\bigcup_I S_{\mathcal{D},I} \right). \quad (4)$$

into strata of the form $S_{\mathcal{D},I}$. A choice of ordering of the simple objects of \mathcal{D}^0 determines a homeomorphism $S_{\mathcal{D}} \cong (\mathbb{H}_-)^n$ under which the decomposition into strata corresponds to the the apparent decomposition of $(\mathbb{H}_-)^n$ with $S_{\mathcal{D},I} \cong \mathbb{H}^{n-\#I} \times \mathbb{R}_{<0}^{\#I}$ where \mathbb{H} is the strict upper half plane in \mathbb{C} . In particular each stratum $S_{\mathcal{D},I}$ is contractible.

Consider the closure $\overline{S_{\mathcal{D},I}}$ of a stratum. For $I \subseteq K \subseteq \{s_1, \dots, s_n\}$ let

$$\partial_K S_{\mathcal{D},I} = \{\sigma \in \overline{S_{\mathcal{D},I}} : \text{Im } Z_{\sigma}(s) = 0 \iff s \in K\},$$

so that $\overline{S_{\mathcal{D},I}} = \bigsqcup_K \partial_K S_{\mathcal{D},I}$ (as a set). For example $\partial_I S_{\mathcal{D},I} = S_{\mathcal{D},I}$.

Lemma 3.9. *For any t-structure \mathcal{E} , not necessarily algebraic, the intersection $S_{\mathcal{E}} \cap \partial_K S_{\mathcal{D},I}$ is a union of components of $\partial_K S_{\mathcal{D},I}$, i.e. the heart of the stability condition remains constant in each component of $\partial_K S_{\mathcal{D},I}$. Each such component which lies in $\text{Stab}_{\text{alg}}(\mathcal{C})$ is a stratum $S_{\mathcal{E},J}$ for some \mathcal{E} and subset J of the simple objects in \mathcal{E} , with $\#J = \#K$.*

Proof. Suppose $\sigma_n \rightarrow \sigma$ in $\text{Stab}(\mathcal{C})$. Then $\mathcal{P}_{\sigma}(0) = \langle 0 \neq c \in \mathcal{C} : \varphi_n^{\pm}(c) \rightarrow 0 \rangle$ by [53, §3]. If $\sigma_n \in S_{\mathcal{D}}$ for all n then

$$\mathcal{P}_{\sigma}(0) = \left\langle \{0 \neq d \in \mathcal{D}^0 : \varphi_n^+(d) \rightarrow 0\}, \{0 \neq d \in \mathcal{D}^0 : \varphi_n^-(d) \rightarrow 1\}[-1] \right\rangle.$$

Furthermore, \mathcal{D}_{σ} is the right tilt of \mathcal{D} at the torsion theory

$$\left\langle 0 \neq d \in \mathcal{D}^0 : \varphi_n^-(d) \not\rightarrow 0 \right\rangle = {}^{\perp} \left\langle 0 \neq d \in \mathcal{D}^0 : \varphi_n^+(d) \rightarrow 0 \right\rangle. \quad (5)$$

Now suppose $\sigma \in \partial_K S_{\mathcal{D},I}$ and (σ_n) is a sequence in $S_{\mathcal{D},I}$ with limit σ . If $\varphi_n^+(d) \rightarrow 0$ for some $0 \neq d \in \mathcal{D}^0$ then $Z_n(d) \rightarrow Z_{\sigma}(d) \in \mathbb{R}_{>0}$. Hence $d \in \langle K \rangle$. For $d \in \langle K \rangle$ there are three possibilities:

- (1) $\varphi_n^{\pm}(d) \rightarrow 0$ and $d \in \mathcal{P}_{\sigma}(0)$;
- (2) $\varphi_n^{\pm}(d) \rightarrow 1$ and $d \in \mathcal{P}_{\sigma}(1)$;
- (3) $\varphi_n^-(d) \rightarrow 0$, $\varphi_n^+(d) \rightarrow 1$, and d is not σ -semistable.

Since the upper and lower phases of d are continuous in $\text{Stab}(\mathcal{C})$, and the possibilities are distinguished by discrete conditions on the limiting phases, we deduce that the torsion theory (5) is constant for σ in a component of $\partial_K S_{\mathcal{D},I}$. Hence the component is contained in $S_{\mathcal{E}}$ for some t-structure \mathcal{E} , and $S_{\mathcal{E}} \cap \partial_K S_{\mathcal{D},I}$ is a union of components of $\partial_K S_{\mathcal{D},I}$ as claimed.

Now suppose that $\sigma \in S_{\mathcal{E},J} \cap \partial_K S_{\mathcal{D},I}$ for some algebraic \mathcal{E} . On the one hand, $\langle J \rangle = \mathcal{P}_\sigma(1)$ since $\sigma \in S_{\mathcal{E},J}$, and therefore the triangulated closure of J is $\mathcal{P}_\sigma(\mathbb{Z}) = \langle \mathcal{P}_\sigma(\varphi) : \varphi \in \mathbb{Z} \rangle$. On the other hand, $\sigma \in \partial_K S_{\mathcal{D},I}$ implies that $\mathcal{P}_\sigma(\mathbb{Z})$ is also the triangulated closure of the set K of simple objects. The image of the map on Grothendieck groups induced by the inclusion $\mathcal{P}_\sigma(\mathbb{Z}) \hookrightarrow \mathcal{C}$ is therefore $\langle [t] : t \in J \rangle = \langle [s] : s \in K \rangle$. Since the elements of J are simple objects in the heart of \mathcal{E} , and those of K are simple objects in the heart of \mathcal{D} , and both \mathcal{D} and \mathcal{E} are algebraic by assumption, this is a free subgroup of rank $\#J = \#K$.

By a similar argument to that used for the first part of this proof

$$\left\langle 0 \neq d \in \mathcal{D}^0 : \varphi_n^-(d) \rightarrow 1 \right\rangle$$

is constant for σ in a component of $\partial_K S_{\mathcal{D},I}$. It follows that $\mathcal{P}_\sigma(0)$ is constant in a component. By the first part \mathcal{E} is fixed by the choice of component. As $\langle J \rangle = \mathcal{P}_\sigma(1) = \mathcal{P}_\sigma(0)[1]$ the subset J of simple objects in \mathcal{E} is also fixed. So each component A of $\text{Stab}_{\text{alg}}(\mathcal{C}) \cap \partial_K S_{\mathcal{D},I}$ is contained in some stratum $S_{\mathcal{E},J}$. The fact that we can perturb a stability condition by perturbing the charge allows us to deduce that $\partial_K S_{\mathcal{D},I}$ is a codimension $\#K$ submanifold of $\text{Stab}(\mathcal{C})$ and that $S_{\mathcal{E},J}$ is a codimension $\#J$ submanifold. Since $\#J = \#K$ the component A must be an open subset of $S_{\mathcal{E},J}$. But directly from the definition of $\partial_K S_{\mathcal{D},I}$ one sees that the component A is also a closed subset and, since $S_{\mathcal{E},J}$ is connected, we deduce that $A = S_{\mathcal{E},J}$ as required. \square

Corollary 3.10. *The decomposition (4) of $\text{Stab}_{\text{alg}}(\mathcal{C})$ satisfies the frontier condition, i.e. if $S_{\mathcal{E},J} \cap \overline{S_{\mathcal{D},I}} \neq \emptyset$ then $S_{\mathcal{E},J} \subseteq \overline{S_{\mathcal{D},I}}$. In particular, the closure of each stratum is a union of lower-dimensional strata. Moreover,*

$$S_{\mathcal{E},J} \subseteq \overline{S_{\mathcal{D},I}} \quad \Rightarrow \quad \mathcal{E} \leq \mathcal{D} \leq L_I \mathcal{D} \leq L_J \mathcal{E}.$$

Proof. The frontier condition follows immediately from Lemma 3.9. Suppose that $S_{\mathcal{E},J} \subseteq \overline{S_{\mathcal{D},I}}$, and choose σ in $S_{\mathcal{E},J}$. Let $\sigma_n \rightarrow \sigma$ where $\sigma_n \in S_{\mathcal{D},I}$. Then $\mathcal{D}^{\leq 0} = \mathcal{P}_n(0, \infty)$, $\mathcal{D}_I^{\leq 0} = \mathcal{P}_n[0, \infty)$, $\mathcal{E}^{\leq 0} = \mathcal{P}_\sigma(0, \infty)$, and $\mathcal{E}_J^{\leq 0} = \mathcal{P}_\sigma[0, \infty)$. Since $\mathcal{P}_n(0, \infty)$ and $\mathcal{P}_n[0, \infty)$ do not vary with n , and the minimal phase $\varphi_\tau^-(c)$ of any $0 \neq c \in \mathcal{C}$ is continuous in τ ,

$$\mathcal{P}_\sigma(0, \infty) \subseteq \mathcal{P}_n(0, \infty) \subseteq \mathcal{P}_n[0, \infty) \subseteq \mathcal{P}_\sigma[0, \infty),$$

i.e. $\mathcal{E} \subseteq \mathcal{D} \subseteq L_I \mathcal{D} \subseteq L_J \mathcal{E}$. Since all these t-structures are in the interval between \mathcal{E} and $\mathcal{E}[-1]$ Remark 2.6 implies that $\mathcal{E} \leq \mathcal{D} \leq L_I \mathcal{D} \leq L_J \mathcal{E}$. \square

Lemma 3.11. *Suppose \mathcal{D} and \mathcal{E} are algebraic t-structures, and that I and J are subsets of simple objects in the respective hearts. If $\mathcal{E} \leq \mathcal{D} \leq L_I \mathcal{D} \leq L_J \mathcal{E}$ then $S_{\mathcal{E},J} \subseteq \overline{S_{\mathcal{D},I}}$.*

Proof. Fix $\sigma \in S_{\mathcal{E},J}$. Since $\mathcal{E} \leq \mathcal{D} \leq L_J \mathcal{E}$ we know that $\mathcal{D} = L_{\mathcal{T}} \mathcal{E}$ for some torsion structure \mathcal{T} on \mathcal{E}^0 , and moreover that $\mathcal{T}^{\leq 0} \subseteq \langle J \rangle = \mathcal{P}_\sigma(1)$. Any simple object of \mathcal{D}^0 lies either in $\mathcal{T}^{\leq 0}[-1]$ or in $\mathcal{T}^{\geq 1}$. Hence any simple object s of \mathcal{D}^0 lies in $\mathcal{P}_\sigma[0, 1]$, and $s \in \mathcal{P}_\sigma(0) \iff s \in \mathcal{T}^{\leq 0}[-1]$. Moreover, if $s \in I$ then $s[-1] \in L_I \mathcal{D}^{\leq 0} \subseteq L_J \mathcal{E}^{\leq 0} = \mathcal{P}_\sigma[0, \infty)$. Thus $s \in I \Rightarrow s \in \mathcal{P}_\sigma(1)$.

Since the simple objects of \mathcal{D}^0 form a basis of $K\mathcal{C}$ we can perturb σ by perturbing their charges. Given $\delta > 0$ we can always make such a perturbation to obtain a stability condition τ with $d(\sigma, \tau) < \delta$ for which $Z_\tau(s) \in \mathbb{H} \cup \mathbb{R}_{>0}$

for all simple s in \mathcal{D}^0 , and $Z_\tau(s) \in \mathbb{R}_{>0} \iff s \in \mathcal{P}_\sigma(0)$. We can then rotate, i.e. act by some $\lambda \in \mathbb{R}$, to obtain a stability condition ω with $d(\tau, \omega) < \delta$ such that $Z_\tau(s) \in \mathbb{H}$ for all simple s in \mathcal{D} . We will prove that $\omega \in S_{\mathcal{D}}$. Since the perturbation and rotation can be chosen arbitrarily small it will follow that $\sigma \in \overline{S_{\mathcal{D}}}$. And since $s \in \mathcal{P}_\sigma(1)$ whenever $s \in I$ we can refine this statement to $\sigma \in \overline{S_{\mathcal{D}, I}}$ as claimed.

It remains to prove $\omega \in S_{\mathcal{D}}$. For this it suffices to show that each simple s in \mathcal{D}^0 is τ -semistable. For then s is ω -semistable too, and the choice of Z_ω implies that $s \in \mathcal{P}_\omega(0, 1]$. The hearts of distinct (bounded) t-structures cannot be nested, so this implies $\mathcal{D} = \mathcal{D}_\omega$, or equivalently $\omega \in S_{\mathcal{D}}$ as required.

Since \mathcal{E} is algebraic Lemma 3.1 guarantees that there is some $\delta > 0$ such that $\mathcal{P}_\sigma(0, 2\delta] = \emptyset$. Provided $d(\sigma, \tau) < \delta$ we have

$$\mathcal{P}_\sigma(0, 1] = \mathcal{P}_\sigma(2\delta, 1] \subseteq \mathcal{P}_\tau(\delta, 1 + \delta] \subseteq \mathcal{P}_\sigma(0, 1 + 2\delta] = \mathcal{P}_\sigma(0, 1].$$

It follows that the Harder–Narasimhan τ -filtration of any $e \in \mathcal{E}^0 = \mathcal{P}_\sigma(0, 1]$ is a filtration by subobjects of e in the abelian category $\mathcal{P}_\sigma(0, 1]$.

Consider a simple s' in \mathcal{D}^0 with $s'[1] \in \mathcal{T}^{\leq 0}$. Since $\mathcal{T}^{\leq 0}$ is a torsion theory any quotient of $s'[1]$ is also in $\mathcal{T}^{\leq 0}$, in particular the final factor in the Harder–Narasimhan τ -filtration, t say, is in $\mathcal{T}^{\leq 0}$. Hence $t[-1] \in \mathcal{D}^0$ and $[t] = -\sum m_s[s] \in K\mathcal{C}$ where the sum is over the simple s in \mathcal{D}^0 and the $m_s \in \mathbb{N}$. Since $\text{Im } Z_\tau(s) \geq 0$ for each simple s it follows that $\text{Im } Z_\tau(t) = -\sum m_s \text{Im } Z_\tau(s) \leq 0$. Combined with the fact that t is τ -semistable with phase in $(\delta, 1 + \delta]$ we have $\varphi_\tau^-(s'[1]) = \varphi_\tau(t) \geq 1$. Hence $s' \in \mathcal{P}_\tau[1, 1 + \delta]$. But $s'[1] \in \mathcal{T}^{\leq 0}$ so $Z_\tau(s'[1]) \in \mathbb{R}_{<0}$ and therefore $s'[1] \in \mathcal{P}_\tau(1)$, and in particular is τ -semistable.

Now suppose $s' \in \mathcal{T}^{\geq 1}$. Since $\mathcal{T}^{\geq 1}$ is a torsion-free theory in $\mathcal{P}_\sigma(0, 1]$ any subobject of s' is also in $\mathcal{T}^{\geq 1}$. In contrast, s' cannot have any *proper* quotients in $\mathcal{T}^{\geq 1}$: if it did we would obtain a short exact sequence

$$0 \rightarrow f \rightarrow s \rightarrow f' \rightarrow 0$$

in $\mathcal{P}_\sigma(0, 1]$ with $f, f' \in \mathcal{T}^{\geq 1}$. This would also be short exact in \mathcal{D}^0 , contradicting the fact that s' is simple. It follows that any proper quotient of s' is in $\mathcal{T}^{\leq 0}$. The argument of the previous paragraph then shows that either s' is τ -semistable (with no proper semistable quotient), or $s' \in \mathcal{P}_\tau[1, 1 + \delta]$. But $\text{Im } Z_\tau(s') > 0$ so the latter is impossible, and s' must be τ -semistable. This completes the proof. \square

Definition 3.12. Let $\text{Int}(\mathcal{C})$ be the poset whose elements are intervals in the poset $\text{Tilt}(\mathcal{C})$ of t-structures of the form $[\mathcal{D}, L_I \mathcal{D}]_{\leq}$, where \mathcal{D} is algebraic and I is a subset of the simple objects in the heart of \mathcal{D} . We order these intervals by inclusion. We do not assume that $L_I \mathcal{D}$ is algebraic.

Corollary 3.13. *There is an isomorphism $\text{Int}(\mathcal{C})^{op} \rightarrow P(\text{Stab}_{\text{alg}}(\mathcal{C}))$ of posets given by the correspondence $[\mathcal{D}, L_I \mathcal{D}]_{\leq} \longleftrightarrow S_{\mathcal{D}, I}$. Components of $\text{Stab}_{\text{alg}}(\mathcal{C})$ correspond to components of $\text{Tilt}_{\text{alg}}(\mathcal{C})$.*

Proof. The existence of the isomorphism is direct from Corollary 3.10 and Lemma 3.11. In particular, components of these posets are in 1-to-1 correspondence. The second statement follows because components of $\text{Stab}_{\text{alg}}(\mathcal{C})$ correspond to components of $P(\text{Stab}_{\text{alg}}(\mathcal{C}))$, and components of $\text{Int}(\mathcal{C})$ correspond to components of $\text{Tilt}_{\text{alg}}(\mathcal{C})$. \square

Remark 3.14. Following Remark 2.8 we note an alternative description of $\text{Int}(\mathcal{C})$ when $\mathcal{C} = \mathcal{D}(A)$ is the bounded derived category of a finite-dimensional algebra A over an algebraically-closed field, and has finite global dimension. By [19, Lemma 4.1] $\text{Int}(\mathcal{C})^{\text{op}} \cup \{\hat{0}\} \cong \mathbb{P}_2(\mathcal{C})$ is the poset of silting pairs defined in [19, §3], where $\hat{0}$ is a formally adjoined minimal element. Hence, by the above corollary, $P(\text{Stab}_{\text{alg}}(\mathcal{C})) \cup \{\hat{0}\} \cong \mathbb{P}_2(\mathcal{C})$.

Remark 3.15. If \mathcal{D} and \mathcal{E} are not both algebraic then $\mathcal{D} \leq \mathcal{E} \leq \mathcal{D}[-1]$ need not imply $S_{\mathcal{D}} \cap \overline{S_{\mathcal{E}}} \neq \emptyset$, see [52, p20] for an example. Thus components of $\text{Stab}_{\text{alg}}(\mathcal{C})$ may not correspond to components of $\text{Tilt}(\mathcal{C})$. In general we have maps

$$\begin{array}{ccccc} \pi_0 \text{Stab}_{\text{alg}}(\mathcal{C}) & \longrightarrow & \pi_0 \text{Stab}(\mathcal{C}) & & \\ \parallel & & \downarrow & & \\ \pi_0 \text{Tilt}_{\text{alg}}(\mathcal{C}) & \longrightarrow & \pi_0 \text{Tilt}(\mathcal{C}) & \longrightarrow & \pi_0 \text{T}(\mathcal{C}). \end{array}$$

The bottom row is induced from the maps $\text{Tilt}_{\text{alg}}(\mathcal{C}) \rightarrow \text{Tilt}(\mathcal{C}) \rightarrow \text{T}(\mathcal{C})$, the vertical equality holds by the above corollary, and the vertical map exists because $S_{\mathcal{D}}$ and $S_{\mathcal{E}}$ in the same component of $\text{Stab}(\mathcal{C})$ implies that \mathcal{D} and \mathcal{E} are related by a finite sequence of tilts [53, Corollary 5.2].

Lemma 3.16. Suppose that $\text{Tilt}_{\text{alg}}(\mathcal{C}) = \text{Tilt}(\mathcal{C}) = \text{T}(\mathcal{C})$ are non-empty. Then $\text{Stab}_{\text{alg}}(\mathcal{C}) = \text{Stab}(\mathcal{C})$ has a single component.

Proof. It is clear that $\text{Stab}(\mathcal{C}) = \text{Stab}_{\text{alg}}(\mathcal{C}) \neq \emptyset$. Let $\sigma, \tau \in \text{Stab}(\mathcal{C})$. Since $\text{Tilt}_{\text{alg}}(\mathcal{C}) = \text{Tilt}(\mathcal{C})$ the associated t-structures \mathcal{D}_{σ} and \mathcal{D}_{τ} are algebraic, so that $\mathcal{D}_{\sigma} \subseteq \mathcal{D}_{\tau}[-j]$ for some $j \in \mathbb{N}$ by Lemma 2.9. Since $\text{Tilt}_{\text{alg}}(\mathcal{C}) = \text{T}(\mathcal{C})$ this implies $\mathcal{D}_{\sigma} \preceq \mathcal{D}_{\tau}[-j]$, and thus \mathcal{D}_{σ} and \mathcal{D}_{τ} are in the same component of $\text{Tilt}_{\text{alg}}(\mathcal{C})$. Hence by Corollary 3.13 σ and τ are in the same component of $\text{Stab}_{\text{alg}}(\mathcal{C}) = \text{Stab}(\mathcal{C})$. \square

Lemma 3.17. Suppose $\mathcal{C} = \mathcal{D}(A)$ for a finite-dimensional algebra A over an algebraically-closed field, with finite global dimension. Then $\text{Stab}_{\text{alg}}(\mathcal{C})$ is connected. Moreover, any component of $\text{Stab}(\mathcal{C})$ other than that containing $\text{Stab}_{\text{alg}}(\mathcal{C})$ consists entirely of stability conditions for which the phases of semistable objects are dense in \mathbb{R} .

Proof. By Remark 2.8 $\text{Tilt}_{\text{alg}}(\mathcal{C})$ is the sub-poset of $\text{T}(\mathcal{C})$ consisting of the algebraic t-structures. The proof that $\text{Stab}_{\text{alg}}(\mathcal{C})$ is connected is then the same as that of the previous result. For the last part note that if σ is a stability condition for which the phases of semistable objects are not dense then acting on σ by some element of \mathbb{C} we obtain an algebraic stability condition. Hence σ must be in the unique component of $\text{Stab}(\mathcal{C})$ containing $\text{Stab}_{\text{alg}}(\mathcal{C})$. \square

Remark 3.18. To show that $\text{Stab}(\mathcal{C})$ is connected when $\mathcal{C} = \mathcal{D}(A)$ as in the previous result it suffices to show that there are no stability conditions for which the phases of semistable objects are dense. For example, from Example 3.5, and the fact that the path algebra of an acyclic quiver is a finite-dimensional algebra of global dimension 1, we conclude that $\text{Stab}(Q)$ is connected whenever Q is of ADE Dynkin, or extended Dynkin, type. (Later

we show that $\text{Stab}(Q)$ is contractible in the Dynkin case; it was already known to be simply-connected by [43].)

By Remark 3.6, the universal cover $G = \widetilde{GL_2^+(\mathbb{R})}$ acts freely on a component consisting of stability conditions for which the phases are dense. In contrast, it does not act freely on a component containing algebraic stability conditions since any such contains stability conditions for which the central charge is real, and these have non-trivial stabiliser. Hence, the G -action also distinguishes the component containing $\text{Stab}_{\text{alg}}(\mathcal{C})$ from the others, and if there is no component on which G acts freely $\text{Stab}(\mathcal{C})$ must be connected.

Suppose $\text{Stab}_{\text{alg}}(\mathcal{C}) \neq \emptyset$. Let $\text{Bases}(K\mathcal{C})$ be the groupoid whose objects are pairs consisting of an ordered basis of the free abelian group $K\mathcal{C}$ and a subset of this basis, and whose morphisms are automorphisms relating these bases (so there is precisely one morphism in each direction between any two objects; we do not ask that it preserve the subsets). Fix an ordering of the simple objects in the heart of each algebraic t-structure. This fixes isomorphisms

$$S_{\mathcal{D},I} \cong \mathbb{H}^{n-\#I} \times \mathbb{R}_{<0}^{\#I}.$$

Regard the poset $\text{Int}(\mathcal{C})$ as a category, and let $F_{\mathcal{C}}: \text{Int}(\mathcal{C}) \rightarrow \text{Bases}(K\mathcal{C})$ be the functor taking $[\mathcal{D}, L_I\mathcal{D}]_{\leq}$ to the pair consisting of the ordered basis of classes of simple objects in \mathcal{D} and the subset of classes of I . This uniquely specifies $F_{\mathcal{C}}$ on morphisms.

Proposition 3.19. *The functor $F_{\mathcal{C}}$ determines $\text{Stab}_{\text{alg}}(\mathcal{C})$ up to homeomorphism as a space over $\text{Hom}(K\mathcal{C}, \mathbb{C})$.*

Proof. As sets there is a commutative diagram

$$\begin{array}{ccc} \text{Stab}_{\text{alg}}(\mathcal{C}) & \xrightarrow{\beta} & \sum_{\mathcal{D},I} \mathbb{H}^{n-\#I} \times \mathbb{R}_{<0}^{\#I} \\ & \searrow \pi & \swarrow \sum \pi_{\mathcal{D},I} \\ & \text{Hom}(K\mathcal{C}, \mathbb{C}) & \end{array}$$

where the map $\pi_{\mathcal{D},I}$ is determined from the pair $F_{\mathcal{C}}([\mathcal{D}, L_I\mathcal{D}]_{\leq})$ of basis and subset, and β is defined using the bijections $S_{\mathcal{D},I} \cong \mathbb{H}^{n-\#I} \times \mathbb{R}_{<0}^{\#I}$. The subsets

$$U_{\mathcal{E},J} = \bigcup_{\mathcal{E} \leq \mathcal{D} \leq L_I\mathcal{D} \leq L_J\mathcal{E}} \pi_{\mathcal{D},I}^{-1}U,$$

where U is open in $\text{Hom}(K\mathcal{C}, \mathbb{C})$, form a base for a topology. With this topology, β is a homeomorphism. To see this note that

$$\beta^{-1}U_{\mathcal{E},J} = \left(\bigcup_{\mathcal{E} \leq \mathcal{D} \leq L_I\mathcal{D} \leq L_J\mathcal{E}} S_{\mathcal{D},I} \right) \cap \pi^{-1}U$$

is the intersection of an open subset with an upward-closed union of strata, hence open. So β is continuous. Moreover, all sufficiently small open neighbourhoods of a point of $\text{Stab}_{\text{alg}}(\mathcal{C})$ have this form, so the bijection β is an open map, hence a homeomorphism. \square

847 A more practical approach is to study the homotopy-type of $\text{Stab}_{\text{alg}}(\mathcal{C})$.
 848 In good cases this is encoded in the poset $P(\text{Stab}_{\text{alg}}(\mathcal{C})) \cong \text{Int}(\mathcal{C})^{\text{op}}$.

849 Recall that a stratification is *locally-finite* if any stratum is contained in
 850 the closure of only finitely many other strata, and *closure-finite* if the closure
 851 of each stratum is a union of finitely many strata.

852 **Lemma 3.20.** *The following are equivalent:*

- 853 (1) *the stratification of $\text{Stab}_{\text{alg}}(\mathcal{C})$ is locally-finite;*
- 854 (2) *the stratification of $\text{Stab}_{\text{alg}}(\mathcal{C})$ is closure-finite;*
- 855 (3) *each interval $[\mathcal{D}, \mathcal{D}[-1]]_{\preceq}$ in $\text{Tilt}_{\text{alg}}(\mathcal{C})$ is finite.*

856 *Proof.* This follows easily from Corollary 3.13 which states that $S_{\mathcal{E},J} \subseteq$
 857 $\overline{S_{\mathcal{D},I}} \iff \mathcal{E} \leq \mathcal{D} \leq L_I \mathcal{D} \leq L_J \mathcal{E}$. Thus the size of the interval $[\mathcal{D}, \mathcal{D}[-1]]_{\preceq}$
 858 is precisely

$$\#\{\mathcal{E} \in \text{Tilt}_{\text{alg}}(\mathcal{C}) : \overline{S_{\mathcal{E}}} \cap S_{\mathcal{D}} \neq \emptyset\} = \#\{\mathcal{E} \in \text{Tilt}_{\text{alg}}(\mathcal{C}) : \overline{S_{\mathcal{D}}} \cap S_{\mathcal{E}[1]} \neq \emptyset\}.$$

859 The result follows because each $S_{\mathcal{D}}$ is a finite union of strata, and each
 860 stratum is in some $S_{\mathcal{D}}$. \square

861 **Proposition 3.21.** *The space $\text{Stab}_{\text{alg}}(\mathcal{C})$ of algebraic stability conditions,*
 862 *with the decomposition into the strata $S_{\mathcal{D},I}$, can be given the structure of*
 863 *a regular, normal cellular stratified space. It is a regular, totally-normal*
 864 *CW-cellular stratified space precisely when $\text{Stab}_{\text{alg}}(\mathcal{C})$ is locally-finite.*

865 *Proof.* First we define a cell structure on $S_{\mathcal{D},I}$. Denote the projection onto
 866 the central charge by $\pi: \text{Stab}(\mathcal{C}) \rightarrow \text{Hom}(K\mathcal{C}, \mathbb{C})$. Choose a basis for $K\mathcal{C}$
 867 and identify $\text{Hom}(K\mathcal{C}, \mathbb{C}) \cong \mathbb{C}^n \cong \mathbb{R}^{2n}$ with $2n$ -dimensional Euclidean space.
 868 Note that

$$\overline{S_{\mathcal{D},I}} \cap \text{Stab}_{\text{alg}}(\mathcal{C}) \cong \pi(\overline{S_{\mathcal{D},I}} \cap \text{Stab}_{\text{alg}}(\mathcal{C})) \subseteq \overline{\pi(S_{\mathcal{D},I})}$$

869 and that $\overline{\pi(S_{\mathcal{D},I})}$ is the real convex closed polyhedral cone

$$C = \{Z : \text{Im } Z(s) \geq 0 \text{ for } s \notin I \text{ and } \text{Im } Z(s) = 0, \text{Re } Z(s) \leq 0 \text{ for } s \in I\}$$

870 in $\text{Hom}(K\mathcal{C}, \mathbb{C})$. The projection π identifies the stratum $S_{\mathcal{D},I}$ with the (rel-
 871 ative) interior of C . By Corollary 3.10 $\overline{S_{\mathcal{D},I}} \cap \text{Stab}_{\text{alg}}(\mathcal{C})$ is a union of strata.
 872 Moreover, the projection of each boundary stratum

$$S_{\mathcal{E},J} \subseteq \overline{S_{\mathcal{D},I}} \cap \text{Stab}_{\text{alg}}(\mathcal{C})$$

873 is cut out by a finite set of (real) linear equalities and inequalities. Therefore
 874 we can subdivide C into a union of real convex polyhedral sub-cones in such
 875 a way that each stratum is identified with the (relative) interior of one of
 876 these sub-cones.

877 Let $A(1,2)$ be the open annulus in $\text{Hom}(K\mathcal{C}, \mathbb{C})$ consisting of points of
 878 distance in the range $(1,2)$ from the origin, and $A[1,2]$ its closure. Then we
 879 have a continuous map

$$\overline{S_{\mathcal{D},I}} \cap \text{Stab}_{\text{alg}}(\mathcal{C}) \xrightarrow{\pi} C - \{0\} \cong C \cap A(1,2) \hookrightarrow C \cap A[1,2]$$

880 where $C - \{0\}$ is identified with $C \cap A(1,2)$ via a radial contraction. The
 881 subdivision of C into cones induces the structure of a compact curvilinear
 882 polyhedron on the intersection $C \cap A[1,2]$. A choice of homeomorphism from
 883 $C \cap A[1,2]$ to a closed cell yields a map from $\overline{S_{\mathcal{D},I}} \cap \text{Stab}_{\text{alg}}(\mathcal{C})$ to a closed cell

884 which is a homeomorphism onto its image. The inverse from this image is
 885 a characteristic map for the stratum $S_{\mathcal{D},I}$, and the collection of these gives
 886 $\text{Stab}_{\text{alg}}(\mathcal{C})$ the structure of a regular, normal cellular stratified space.

887 When the stratification of $\text{Stab}_{\text{alg}}(\mathcal{C})$ is locally-finite the cellular stratifica-
 888 tion is closure-finite by Lemma 3.20, and any point is contained in the inte-
 889 rior of a closed union of finitely many cells. This guarantees that $\text{Stab}_{\text{alg}}(\mathcal{C})$
 890 has the weak topology arising from the cellular stratification, which is there-
 891 fore a CW-cellular stratification. We can also choose the above subdivision
 892 of C to have finitely many sub-cones. In this case the curvilinear polyhe-
 893 dron $C \cap A[1, 2]$ has finitely many faces, and therefore has a CW-structure
 894 for which the strata of $\overline{S_{\mathcal{D},I}} \cap \text{Stab}_{\text{alg}}(\mathcal{C})$ are identified with certain open
 895 cells. It follows that the cellular stratification is totally-normal. Converse-
 896 ly, if the stratification is CW-cellular then it is closure-finite, and hence by
 897 Lemma 3.20 it is locally-finite. \square

898 **Corollary 3.22.** *Suppose the stratification of $\text{Stab}_{\text{alg}}(\mathcal{C})$ is locally-finite and*
 899 *let $n = \text{rank}(KC)$. Then we have the following:*

- 900 (1) *There is a homotopy equivalence $\text{Stab}_{\text{alg}}(\mathcal{C}) \simeq BP(\text{Stab}_{\text{alg}}(\mathcal{C}))$.*
- 901 (2) *$BP(\text{Stab}_{\text{alg}}(\mathcal{C}))$ is a CW-complex of dimension $\leq n$*
- 902 (3) *The integral homology groups $H_i(\text{Stab}_{\text{alg}}(\mathcal{C})) = 0$ for $i > n$.*

903 *Proof.* The first claim is direct from Proposition 3.21 and Theorem 2.17.
 904 By Corollary 3.22 $\text{Stab}_{\text{alg}}(\mathcal{C}) \simeq BP(\text{Stab}_{\text{alg}}(\mathcal{C}))$. A chain in the poset
 905 $P(\text{Stab}_{\text{alg}}(\mathcal{C}))$ consists of a sequence of strata of $\text{Stab}_{\text{alg}}(\mathcal{C})$ of decreasing
 906 codimension, each in the closure of the next. Since the maximum codimen-
 907 sion of any stratum is n , the length of any chain is less than or equal to
 908 n . Hence $BP(\text{Stab}_{\text{alg}}(\mathcal{C}))$ is a CW-complex of dimension $\leq n$, and the last
 909 claim also follows. \square

910 **Remark 3.23.** If $\text{Stab}_{\text{alg}}(\mathcal{C})$ is locally-finite then any union U of strata of
 911 $\text{Stab}_{\text{alg}}(\mathcal{C})$ is a regular, totally-normal CW-cellular stratified space. Hence
 912 there is a homotopy equivalence $U \simeq BP(U)$ and $H_i(U) = 0$ for $i > n =$
 913 $\text{rank}(KC)$.

914 **Example 3.24.** We continue Example 3.8. The ‘Kronecker heart’

$$\langle \mathcal{O}, \mathcal{O}(-1)[1] \rangle$$

915 of $\mathcal{D}(\mathbb{P}^1)$ is algebraic. There are infinitely many torsion structures on this
 916 heart such that the tilt is a t-structure with heart isomorphic to the Kro-
 917 necker heart [52, §3.2]. It quickly follows from Corollary 3.13 that the strat-
 918 ification of $\text{Stab}_{\text{alg}}(\mathbb{P}^1)$ is neither closure-finite nor locally-finite — see [52,
 919 Figure 5] for a diagram of the codimension 2 strata in the closure of the
 920 stratum corresponding to the Kronecker heart.

921 **3.2. More on the poset of strata.** Corollary 3.22 shows that if $\text{Stab}_{\text{alg}}(\mathcal{C})$
 922 is closure-finite and locally-finite, then its homotopy-theoretic properties are
 923 encoded in the poset $P(\text{Stab}_{\text{alg}}(\mathcal{C}))$. In the remainder of this section we
 924 elucidate some of the latter’s good properties.

925 The assumptions that $\text{Stab}_{\text{alg}}(\mathcal{C})$ is locally-finite and closure-finite are
 926 respectively equivalent to the statements that the unbounded closed intervals
 927 $[S, \infty)$ and $(-\infty, S]$ are finite for each $S \in P(\text{Stab}_{\text{alg}}(\mathcal{C}))$. It follows of

course that closed bounded intervals are also finite, but in fact the latter holds without these assumptions.

Lemma 3.25. *Suppose $S_{\mathcal{E},J} \subseteq \overline{S_{\mathcal{D},I}}$. Then the closed interval $[S_{\mathcal{E},J}, S_{\mathcal{D},I}]$ in $P(\text{Stab}_{\text{alg}}(\mathcal{C}))$ is isomorphic to a sub-poset of $[I, K]^{\text{op}}$. Here the subset K is uniquely determined by the requirement that $S_{\mathcal{E},J} \subseteq \partial_K S_{\mathcal{D},I}$, and subsets of the simple objects in \mathcal{D}^0 are ordered by inclusion.*

Proof. Suppose $S_{\mathcal{E},J} \subseteq \partial_K S_{\mathcal{D},I}$ and fix $\sigma \in S_{\mathcal{E},J}$. Using the fact that $\text{Stab}(\mathcal{C})$ is locally isomorphic to $\text{Hom}(K\mathcal{C}, \mathbb{C})$ we can choose an open neighbourhood U of σ in $\text{Stab}(\mathcal{C})$ so that $U \cap \partial_L S_{\mathcal{D},I}$ is non-empty and connected for any subset $I \subseteq L \subseteq K$, and empty when $L \not\subseteq K$. It follows that U meets a unique component of $\partial_L S_{\mathcal{D},I}$ for each $I \subseteq L \subseteq K$. The strata in $[S_{\mathcal{E},J}, S_{\mathcal{D},I}]$ correspond to those components for which the heart is algebraic. Since $\partial_L S_{\mathcal{D},I} \subseteq \overline{\partial_{L'} S_{\mathcal{D},I}} \iff L' \subseteq L$ the result follows. \square

We have seen that $\text{Stab}_{\text{alg}}(\mathcal{C})$ need be neither open nor closed as a subset of $\text{Stab}(\mathcal{C})$. The next two results show that whether or not it is locally closed is closely related to the structure of the bounded closed intervals in $P(\text{Stab}_{\text{alg}}(\mathcal{C}))$.

Lemma 3.26. *The first of the statements below implies the second and third, which are equivalent. When $\text{Stab}_{\text{alg}}(\mathcal{C})$ is locally-finite all three are equivalent.*

- (1) *The subset $\text{Stab}_{\text{alg}}(\mathcal{C})$ is locally closed as a subspace of $\text{Stab}(\mathcal{C})$.*
- (2) *The inclusion $\text{Stab}_{\text{alg}}(\mathcal{C}) \cap \overline{S_{\mathcal{D}}} \hookrightarrow \overline{S_{\mathcal{D}}}$ is open for each algebraic \mathcal{D} .*
- (3) *For each pair of strata $S_{\mathcal{E},J} \subseteq \overline{S_{\mathcal{D},I}}$ there is an isomorphism*

$$[S_{\mathcal{E},J}, S_{\mathcal{D},I}] \cong [I, K]^{\text{op}},$$

where K is uniquely determined by the requirement that $S_{\mathcal{E},J} \subseteq \partial_K S_{\mathcal{D},I}$.

Proof. Suppose $\text{Stab}_{\text{alg}}(\mathcal{C})$ is locally closed. Let $\sigma \in \text{Stab}_{\text{alg}}(\mathcal{C}) \cap \overline{S_{\mathcal{D}}}$ where \mathcal{D} is algebraic. Then there is a neighbourhood U of σ in $\text{Stab}(\mathcal{C})$ such that $U \cap \text{Stab}_{\text{alg}}(\mathcal{C})$ is closed in U . Then $U \cap S_{\mathcal{D}} \subseteq U \cap \text{Stab}_{\text{alg}}(\mathcal{C})$ so

$$U \cap \overline{S_{\mathcal{D}}} \subseteq U \cap \text{Stab}_{\text{alg}}(\mathcal{C})$$

and $\text{Stab}_{\text{alg}}(\mathcal{C}) \cap \overline{S_{\mathcal{D}}}$ is open in $\overline{S_{\mathcal{D}}}$.

Now suppose $\text{Stab}_{\text{alg}}(\mathcal{C}) \cap \overline{S_{\mathcal{D}}}$ is open in $\overline{S_{\mathcal{D}}}$. Then we can choose a neighbourhood U of σ so that $U \cap \partial_L S_{\mathcal{D},I}$ is non-empty and connected for each $I \subseteq L \subseteq K$ and, moreover, $U \cap \overline{S_{\mathcal{D}}} \subseteq \text{Stab}_{\text{alg}}(\mathcal{C})$. It follows, as in the proof of Lemma 3.25, that $[S_{\mathcal{E},J}, S_{\mathcal{D},I}] \cong [I, K]^{\text{op}}$.

Conversely, if $[S_{\mathcal{E},J}, S_{\mathcal{D},I}] \cong [I, K]^{\text{op}}$ then given a neighbourhood U with $U \cap \partial_L S_{\mathcal{D},I}$ non-empty and connected for each $I \subseteq L \subseteq K$ we see that it meets only components of the $\partial_L S_{\mathcal{D},I}$ which are in $\text{Stab}_{\text{alg}}(\mathcal{C})$. Hence $\text{Stab}_{\text{alg}}(\mathcal{C}) \cap \overline{S_{\mathcal{D}}}$ is open in $\overline{S_{\mathcal{D}}}$.

Finally, assume the stratification of $\text{Stab}_{\text{alg}}(\mathcal{C})$ is locally-finite and that $\text{Stab}_{\text{alg}}(\mathcal{C}) \cap \overline{S_{\mathcal{D}}} \hookrightarrow \overline{S_{\mathcal{D}}}$ is open for each algebraic \mathcal{D} . Fix $\sigma \in \text{Stab}_{\text{alg}}(\mathcal{C})$. There are finitely many algebraic \mathcal{D} with $\sigma \in \overline{S_{\mathcal{D}}}$. There is an open neighbourhood U of σ in $\text{Stab}(\mathcal{C})$ such that

$$U \cap \overline{S_{\mathcal{D}}} \subseteq \overline{S_{\mathcal{D}}} \cap \text{Stab}_{\text{alg}}(\mathcal{C})$$

for any algebraic \mathcal{D} (the left-hand side is empty for all but finitely many such). Hence

$$U \cap \text{Stab}_{\text{alg}}(\mathcal{C}) = U \cap \bigcup_{\mathcal{D} \text{ alg}} S_{\mathcal{D}} \subseteq U \cap \bigcup_{\mathcal{D} \text{ alg}} \overline{S_{\mathcal{D}}} = \bigcup_{\mathcal{D} \text{ alg}} U \cap \overline{S_{\mathcal{D}}} \subseteq U \cap \text{Stab}_{\text{alg}}(\mathcal{C})$$

and so $U \cap \text{Stab}_{\text{alg}}(\mathcal{C}) = \bigcup_{\mathcal{D} \text{ alg}} U \cap \overline{S_{\mathcal{D}}}$. The latter is a *finite* union of closed subsets of U , hence closed in U . Therefore each $\sigma \in \text{Stab}_{\text{alg}}(\mathcal{C})$ has an open neighbourhood $U \ni \sigma$ such that $U \cap \text{Stab}_{\text{alg}}(\mathcal{C})$ is closed in U . It follows that $\text{Stab}_{\text{alg}}(\mathcal{C})$ is locally closed. \square

Corollary 3.27. *Suppose $\text{Stab}_{\text{alg}}(\mathcal{C})$ is locally closed as a subspace of $\text{Stab}(\mathcal{C})$. Then $P(\text{Stab}_{\text{alg}}(\mathcal{C}))$ is pure of length $n = \text{rank}(KC)$.*

Proof. The stratum $S_{\mathcal{D}, I}$ contains $S_{\mathcal{D}, \{s_1, \dots, s_n\}}$ in its closure, and is in the closure of $S_{\mathcal{D}, \emptyset}$. It follows that any maximal chain in $P(\text{Stab}_{\text{alg}}(\mathcal{C}))$ is in a closed interval of the form $[S_{\mathcal{D}, \{s_1, \dots, s_n\}}, S_{\mathcal{E}, \emptyset}]$. As $\text{Stab}(\mathcal{C})$ is locally closed this is isomorphic to the poset of subsets of an n -element set by Lemma 3.26. This implies $P(\text{Stab}_{\text{alg}}(\mathcal{C}))$ is pure of length n . \square

Example 3.28. Recall Examples 3.8 and 3.24. The subspace $\text{Stab}_{\text{alg}}(\mathbb{P}^1)$ is not locally closed: if it were then $\text{Stab}(\mathbb{P}^1) - \text{Stab}_{\text{alg}}(\mathbb{P}^1) = A \cup U$ for some closed A and open U . This subset consists of those stability conditions for which the phases of semistable objects accumulate at $\mathbb{Z} \subseteq \mathbb{R}$, and this has empty interior. Hence the only possibility is that $U = \emptyset$, in which case $\text{Stab}_{\text{alg}}(\mathbb{P}^1)$ would be open. This is not the case, so $\text{Stab}_{\text{alg}}(\mathbb{P}^1)$ cannot be locally closed. Nevertheless, from the explicit description of stability conditions in [38] one can see that the poset of strata is pure (of rank 2), and that the second two conditions of Lemma 3.26 are satisfied.

4. FINITE-TYPE COMPONENTS

4.1. The main theorem. We say a t-structure is of *finite tilting type* if it is algebraic and has only finitely many torsion-structures in its heart. A t-structure has finite tilting type if and only if it is algebraic and the interval $[\mathcal{D}, \mathcal{D}[-1]]_{\leq}$ in $\text{Tilt}(\mathcal{C})$ is finite. We say a component $\text{Tilt}^{\circ}(\mathcal{C})$ is of *finite tilting type* if each t-structure in it has finite tilting type. It follows from Lemmas 2.13 and 2.14 that a finite tilting type component $\text{Tilt}^{\circ}(\mathcal{C})$ is a lattice, and that closed bounded intervals in it are finite.

Lemma 4.1. *Suppose that the set S of t-structures obtained from some \mathcal{D} by finite sequences of simple tilts consists entirely of t-structures of finite tilting type. Then S is (the underlying set of) a finite tilting type component of $\text{Tilt}(\mathcal{C})$. Moreover, every finite tilting type component arises in this way.*

Proof. If \mathcal{D} has finite tilting type then any tilt of \mathcal{D} can be decomposed into a finite sequence of simple tilts. It follows that S is a component of $\text{Tilt}(\mathcal{C})$ as claimed. It is clearly of finite tilting type. Conversely if $\text{Tilt}^{\circ}(\mathcal{C})$ is a finite tilting type component, and $\mathcal{D} \in \text{Tilt}^{\circ}(\mathcal{C})$, then every t-structure obtained from \mathcal{D} by a finite sequence of simple tilts is algebraic, and has finite tilting type. Hence \mathcal{D} contains the set S , and by the first part $S = \text{Tilt}^{\circ}(\mathcal{C})$. \square

1009 If the heart of a t-structure contains only finitely many isomorphism class-
 1010 es of indecomposable objects, then it is of finite tilting type (because a
 1011 torsion theory is determined by the indecomposable objects it contains).
 1012 Therefore, whilst we do not use it in this paper, the following result may be
 1013 useful in detecting finite tilting type components, particularly if up to au-
 1014 tomorphism there are only finitely many t-structures which can be reached
 1015 from \mathcal{D} by finite sequences of simple tilts. In very good cases — for in-
 1016 stance when tilting at a 2-spherical simple object s with the property that
 1017 $\mathrm{Hom}_{\mathcal{C}}^i(s, s') = 0$ for $i \neq 1$ for any other simple object s' — the tilted t-
 1018 structure itself is obtained by applying an automorphism of \mathcal{C} and hence
 1019 inherits the property of being algebraic of finite tilting type. A similar sit-
 1020 uation arises if \mathcal{D} is an algebraic t-structure in which all simple objects are
 1021 rigid, i.e. have no self extensions. In this case [33, Proposition 5.4] states
 1022 that all simple tilts of \mathcal{D} are also algebraic.

1023 **Lemma 4.2.** *Suppose that \mathcal{D} is a t-structure on a triangulated category \mathcal{C}
 1024 whose heart is a length category with only finitely many isomorphism classes
 1025 of indecomposable objects. Then any simple tilt of \mathcal{D} is algebraic.*

1026 *Proof.* It suffices to prove that the claim holds for any simple right tilt, since
 1027 the simple left tilts are shifts of these. Since there are only finitely many
 1028 indecomposable objects in \mathcal{D}^0 there are in particular only finitely many
 1029 simple objects. Let these be s_1, \dots, s_n and consider the right tilt at s_1 . Let
 1030 $\sigma \in S_{\mathcal{D}}$ be the unique stability condition with $Z_{\sigma}(s_1) = i$ and $Z_{\sigma}(s_j) = -1$
 1031 for $j = 2, \dots, n$. Let τ be obtained by acting on σ by $-1/2 \in \mathbb{C}$. Then \mathcal{D}_{τ}
 1032 is the right tilt of \mathcal{D}_{σ} at s_1 . As there are only finitely many indecomposable
 1033 objects in \mathcal{D}^0 the set of $\varphi \in \mathbb{R}$ such that $\mathcal{P}_{\sigma}(\varphi) \neq \emptyset$ is discrete. The same
 1034 is therefore true for τ . It follows that $\mathcal{P}_{\tau}(0, \epsilon) = \emptyset$ for some $\epsilon > 0$. The
 1035 component of $\mathrm{Stab}(\mathcal{C})$ containing σ and τ is full since σ is algebraic. Hence
 1036 by Lemma 3.1 the stability condition τ is algebraic too. \square

1037 **Lemma 4.3.** *Let $\mathrm{Tilt}^{\circ}(\mathcal{C})$ be a finite tilting type component of $\mathrm{Tilt}(\mathcal{C})$. Then*

$$\mathrm{Stab}^{\circ}(\mathcal{C}) = \bigcup_{\mathcal{D} \in \mathrm{Tilt}^{\circ}(\mathcal{C})} S_{\mathcal{D}} \quad (6)$$

1038 *is a component of $\mathrm{Stab}(\mathcal{C})$.*

1039 *Proof.* Clearly $\mathrm{Tilt}^{\circ}(\mathcal{C})$ is also a component of $\mathrm{Tilt}_{\mathrm{alg}}(\mathcal{C})$. By Corollary 3.13
 1040 there is a corresponding component $\mathrm{Stab}_{\mathrm{alg}}^{\circ}(\mathcal{C})$ of $\mathrm{Stab}_{\mathrm{alg}}(\mathcal{C})$ given by the
 1041 RHS of (6). Let $\mathrm{Stab}^{\circ}(\mathcal{C})$ be the unique component of $\mathrm{Stab}(\mathcal{C})$ containing
 1042 $\mathrm{Stab}_{\mathrm{alg}}^{\circ}(\mathcal{C})$. Recall from [53, Corollary 5.2] that the t-structures associated to
 1043 stability conditions in a component of $\mathrm{Stab}(\mathcal{C})$ are related by finite sequences
 1044 of tilts. Thus, each stability condition in $\mathrm{Stab}^{\circ}(\mathcal{C})$ has associated t-structure
 1045 in $\mathrm{Tilt}^{\circ}(\mathcal{C})$. In particular, the t-structure is algebraic and $\mathrm{Stab}_{\mathrm{alg}}^{\circ}(\mathcal{C}) =$
 1046 $\mathrm{Stab}^{\circ}(\mathcal{C})$ is actually a component of $\mathrm{Stab}(\mathcal{C})$. \square

1047 A *finite-type* component $\mathrm{Stab}^{\circ}(\mathcal{C})$ of $\mathrm{Stab}(\mathcal{C})$ is one which arises in this
 1048 way from a finite tilting type component $\mathrm{Tilt}^{\circ}(\mathcal{C})$ of $\mathrm{Tilt}(\mathcal{C})$.

1049 **Lemma 4.4.** *Suppose $\mathrm{Stab}^{\circ}(\mathcal{C})$ is a finite-type component. The stratifica-
 1050 tion of $\mathrm{Stab}^{\circ}(\mathcal{C})$ is locally-finite and closure-finite.*

1051 *Proof.* This is immediate from Lemma 3.20 and the obvious fact that the interval $[\mathcal{D}_\sigma, \mathcal{D}_\sigma[-1]]_{\preceq}$ of algebraic tilts is finite when the interval $[\mathcal{D}_\sigma, \mathcal{D}_\sigma[-1]]_{\leq}$ of all tilts is finite. \square

1054 **Corollary 4.5.** *Suppose $\text{Stab}^\circ(\mathcal{C})$ is a finite-type component. There is a homotopy equivalence $\text{Stab}^\circ(\mathcal{C}) \simeq BP(\text{Stab}^\circ(\mathcal{C}))$, in particular $\text{Stab}^\circ(\mathcal{C})$ has the homotopy-type of a CW-complex of dimension $\dim_{\mathbb{C}} \text{Stab}^\circ(\mathcal{C})$.*

1057 *Proof.* This is immediate from Lemma 4.4 and Corollary 3.22. \square

1058 We now prove that finite-type components are contractible. Our approach is modelled on the proof of the simply-connectedness of the stability spaces of representations of Dynkin quivers [43, Theorem 4.7]. The key is to show that certain ‘conical unions of strata’ are contractible.

1062 The open star $S_{\mathcal{D},I}^*$ of a stratum $S_{\mathcal{D},I}$ is the union of all strata containing $S_{\mathcal{D},I}$ in their closure. An open star is contractible: $S_{\mathcal{D},I}^* \simeq BP(S_{\mathcal{D},I}^*)$ by Remark 3.23, and, since $P(S_{\mathcal{D},I}^*)$ is a poset with lower bound $S_{\mathcal{D},I}$, its classifying space is contractible.

1066 **Definition 4.6.** For a finite set F of t-structures in $\text{Tilt}^\circ(\mathcal{C})$ let the cone

$$C(F) = \{(\mathcal{E}, J) : \mathcal{F} \preceq \mathcal{E} \preceq L_J \mathcal{E} \preceq \sup F \text{ for some } \mathcal{F} \in F\}.$$

1067 Let $V(F) = \bigcup_{(\mathcal{E}, J) \in C(F)} S_{\mathcal{E}, J}$ be the union of the corresponding strata; we call such a subspace *conical*. For example, $V(\{\mathcal{D}\}) = S_{\mathcal{D}, \emptyset}$. More generally, if $F = \{\mathcal{D}, L_s \mathcal{D} : s \in I\}$ then $\sup F = L_I \mathcal{D}$ and $V(F) = S_{\mathcal{D}, I}^*$.

1070 **Remark 4.7.** If $(\mathcal{E}, J) \in C(F)$ then $\inf F \preceq \mathcal{E} \preceq \sup F$. Since $[\inf F, \sup F]_{\preceq}$ is finite, and there are only finitely many possible J for each \mathcal{E} , it follows that $C(F)$ is a finite set. Let $c(F) = \#C(F)$ be the number of elements, which is also the number of strata in $V(F)$.

1074 Note that $V(F)$ is an open subset of $\text{Stab}^\circ(\mathcal{C})$ since $S_{\mathcal{D}, I} \subseteq V(F)$ and $S_{\mathcal{D}, I} \subseteq \overline{S_{\mathcal{E}, J}}$ implies

$$\mathcal{F} \preceq \mathcal{D} \preceq \mathcal{E} \preceq L_J \mathcal{E} \preceq L_I \mathcal{D} \preceq \sup F$$

1076 for some $\mathcal{F} \in F$ so that $S_{\mathcal{E}, J} \subseteq V(F)$ too. In particular $S_{\mathcal{D}, I} \subseteq V(F)$ implies $S_{\mathcal{D}, I}^* \subseteq V(F)$. It is also non-empty since it contains $S_{\sup F, \emptyset}$.

1078 **Proposition 4.8.** *The conical subspace $V(F)$ is contractible for any finite set $F \subseteq \text{Tilt}^\circ(\mathcal{C})$.*

1080 *Proof.* Let $C = C(F)$, $c = c(F)$, and $V = V(F)$. We prove this result by induction on the number of strata c . When $c = 1$ we have $C = \{(\sup F, \emptyset)\}$ so that $V = S_{\sup F, \emptyset}$ is contractible as claimed. Suppose the result holds for all conical subspaces with strictly fewer than c strata.

1084 Recall from Remark 3.23 that $V \simeq BP(V)$ so that V has the homotopy-type of a CW-complex. Hence it suffices, by the Hurewicz and Whitehead Theorems, to show that V is simply-connected and that the integral homology groups $H_i(V) = 0$ for $i > 0$. Choose $(\mathcal{D}, I) \in C$ such that

- 1088 (1) $\nexists (\mathcal{E}, J) \in C$ with $\mathcal{E} \prec \mathcal{D}$;
- 1089 (2) $(\mathcal{D}, I') \in C \iff I' \subseteq I$.

It is possible to choose such a \mathcal{D} since C is finite; note that \mathcal{D} is necessarily in F . It is then possible to choose such an I because if $S_{\mathcal{D},I'}, S_{\mathcal{D},I''} \subseteq V$ then $L_{I'}\mathcal{D}, L_{I''}\mathcal{D} \preceq \sup F$ which implies $L_{I' \cup I''}\mathcal{D} = L_{I'}\mathcal{D} \vee L_{I''}\mathcal{D} \preceq \sup F$.

The conical subset V has an open cover $V = S_{\mathcal{D},I}^* \cup (V - S_{\mathcal{D}})$. We remarked above that $S_{\mathcal{D},I}^*$ is contractible. In addition, by the choice of \mathcal{D} , the subspace $V - S_{\mathcal{D}} = V(F')$ is also conical, with

$$F' = F \cup \{L_s\mathcal{D} : s \in \mathcal{D}^\circ \text{ simple}, L_s\mathcal{D} \preceq \sup F\} - \{\mathcal{D}\}.$$

Since $V(F')$ has fewer strata than V it is contractible by the inductive hypothesis. Finally, the intersection $S_{\mathcal{D},I}^* \cap (V - S_{\mathcal{D}}) = S_{\mathcal{D},I}^* - S_{\mathcal{D}}$ is the conical subspace

$$\bigcup_{\mathcal{D} \prec \mathcal{E} \preceq L_J \mathcal{E} \preceq L_I \mathcal{D}} S_{\mathcal{E},J} = V(\{L_s\mathcal{D} : s \in I\}),$$

which has fewer strata than V . Hence this too is contractible by the inductive hypothesis. It follows that V is simply-connected by the van Kampen Theorem, and that $H_i(V) = 0$ for $i > 0$ by the Mayer–Vietoris sequence for the open cover by $S_{\mathcal{D},I}^*$ and $V - S_{\mathcal{D}}$. Hence V is contractible by the Hurewicz and Whitehead Theorems. This completes the inductive step. \square

Theorem 4.9. *Suppose $\text{Stab}^\circ(\mathcal{C})$ is a finite-type component. Then $\text{Stab}^\circ(\mathcal{C})$ is contractible.*

Proof. By Lemma 4.4 $\text{Stab}^\circ(\mathcal{C})$ is a locally-finite stratified space. Thus a singular integral i -cycle in $\text{Stab}^\circ(\mathcal{C})$ has support meeting only finitely many strata, say the support is contained in $\{S_{\mathcal{F}} : \mathcal{F} \in F\}$. Therefore the cycle has support in $V(F)$, and so is null-homologous whenever $i > 0$ by Proposition 4.8. This shows that $H_i(\text{Stab}^\circ(\mathcal{C})) = 0$ for $i > 0$. An analogous argument shows that $\text{Stab}^\circ(\mathcal{C})$ is simply-connected. Since $\text{Stab}^\circ(\mathcal{C})$ has the homotopy type of a CW-complex it follows from the Hurewicz and Whitehead Theorems that $\text{Stab}^\circ(\mathcal{C})$ is contractible. \square

We discuss two classes of examples of triangulated categories in which each component of the stability space is of finite-type, and hence is contractible. Each class contains the bounded derived category of finite-dimensional representations of ADE Dynkin quivers, so these can be seen as two ways to generalise from these.

4.2. Locally-finite triangulated categories. We recall the definition of locally-finite triangulated category from [35]. Let \mathcal{C} be a triangulated category. The *abelianisation* $\text{Ab}(\mathcal{C})$ of \mathcal{C} is the full subcategory of functors $F: \mathcal{C}^{\text{op}} \rightarrow \text{Ab}$ fitting into an exact sequence

$$\text{Hom}_{\mathcal{C}}(-, c) \rightarrow \text{Hom}_{\mathcal{C}}(-, c') \rightarrow F \rightarrow 0$$

for some $c, c' \in \mathcal{C}$. The Yoneda embedding $\mathcal{C} \rightarrow \text{Ab}(\mathcal{C})$ is the universal cohomological functor on \mathcal{C} , in the sense that any cohomological functor to an abelian category factors, essentially uniquely, as the Yoneda embedding followed by an exact functor. A triangulated category¹ \mathcal{C} is *locally-finite* if idempotents split and its abelianisation $\text{Ab}(\mathcal{C})$ is a length category. The following ‘internal’ characterisation is due to Auslander [5, Theorem 2.12].

¹Our default assumption that all categories are essentially small is necessary here.

1129 **Proposition 4.10.** *A triangulated category \mathcal{C} in which idempotents are split*
 1130 *is locally-finite if and only if for each $c \in \mathcal{C}$*

- 1131 (1) *there are only finitely many isomorphism classes of indecomposable*
 1132 *objects $c' \in \mathcal{C}$ with $\text{Hom}_{\mathcal{C}}(c', c) \neq 0$;*
 1133 (2) *for each indecomposable $c' \in \mathcal{C}$, the $\text{End}_{\mathcal{C}}(c')$ -module $\text{Hom}_{\mathcal{C}}(c', c)$ has*
 1134 *finite length.*

1135 The category \mathcal{C} is locally-finite if and only if \mathcal{C}^{op} is locally-finite so that
 1136 the above properties are equivalent to the dual ones.

1137 Locally-finite triangulated categories have many good properties: they
 1138 have a Serre functor, equivalently by [45] they have Auslander–Reiten tri-
 1139 angles, the inclusion of any thick subcategory has both left and right ad-
 1140 joints, any thick subcategory, or quotient thereby, is also locally-finite. See
 1141 [35, 3, 54] for further details.

1142 **Lemma 4.11** (cf. [18, Proposition 6.1]). *Suppose that \mathcal{C} is a locally-finite*
 1143 *triangulated category \mathcal{C} with $\text{rank } K\mathcal{C} < \infty$. Then any t-structure on \mathcal{C} is*
 1144 *algebraic, with only finitely many isomorphism classes of indecomposable*
 1145 *objects in its heart.*

1146 *Proof.* Let d be an object in the heart of a t-structure, and suppose it has
 1147 infinitely many pairwise non-isomorphic subobjects. Write each of these
 1148 as a direct sum of the indecomposable objects with non-zero morphisms to
 1149 d . Since there are only finitely many isomorphism classes of such indecom-
 1150 posable objects, there must be one of them, c say, such that $c^{\oplus k}$ appears in
 1151 these decompositions for each $k = 1, 2, \dots$. Hence $c^{\oplus k} \hookrightarrow d$ for each k , which
 1152 contradicts the fact that $\text{Hom}_{\mathcal{C}}(c, d)$ has finite length as an $\text{End}_{\mathcal{C}}(c)$ -module
 1153 (because it has a filtration by $\{\alpha: c \rightarrow d : \alpha \text{ factors through } c^{\oplus k} \rightarrow d\}$ for
 1154 $k \in \mathbb{N}$). We conclude that any object in the heart has only finitely many
 1155 pairwise non-isomorphic subobjects. It follows that the heart is a length
 1156 category. Since $\text{rank } K\mathcal{C} < \infty$ it has finitely many simple objects, and so is
 1157 algebraic.

1158 To see that there are only finitely many indecomposable objects (up to
 1159 isomorphism) note that any indecomposable object in the heart has a simple
 1160 quotient. There are only finitely many such simple objects, and each of these
 1161 admits non-zero morphisms from only finitely many isomorphism classes of
 1162 indecomposable objects. \square

1163 **Remark 4.12.** Since a torsion theory is determined by its indecomposable
 1164 objects it follows that a t-structure on \mathcal{C} as above has only finitely many
 1165 torsion structures on its heart, i.e. it has finite tilting type.

1166 **Corollary 4.13.** *Suppose \mathcal{C} is a locally-finite triangulated category and that*
 1167 *$\text{rank } K\mathcal{C} < \infty$. Then the stability space is a (possibly empty) disjoint union*
 1168 *of finite-type components, each of which is contractible.*

1169 *Proof.* Combining Lemma 4.11 with Lemma 4.1 shows that each compo-
 1170 nent of the tilting poset is of finite tilting type. The result follows from
 1171 Theorem 4.9. \square

1172 **Example 4.14.** Let Q be a quiver whose underlying graph is an ADE
 1173 Dynkin diagram, and suppose the field k is algebraically-closed. Then $\mathcal{D}(Q)$

1174 is a locally-finite triangulated category [30, §2]. The space $\text{Stab}(Q)$ of sta-
 1175 bility conditions is non-empty and connected (by Remark 3.18 or the results
 1176 of [31]), and hence by Corollary 4.13 is contractible. This affirms the first
 1177 part of [43, Conjecture 5.8]. Previously $\text{Stab}(Q)$ was known to be simply-
 1178 connected [43, Theorem 4.7].

1179 **Example 4.15.** For $m \geq 1$ the cluster category $\mathcal{C}_m(Q) = \mathcal{D}(Q)/\Sigma_m$ is
 1180 the quotient of $\mathcal{D}(Q)$ by the automorphism $\Sigma_m = \tau^{-1}[m-1]$, where τ is
 1181 the Auslander–Reiten translation. Each $\mathcal{C}_m(Q)$ is locally-finite [35, §2], but
 1182 $\text{Stab}(\mathcal{C}_m(Q)) = \emptyset$ because there are no t-structures on $\mathcal{C}_m(Q)$.

1183 Remark 5.6 of [43] proposes that $\text{Stab}(\Gamma_N Q) / \text{Br}(\Gamma_N Q)$ should be consid-
 1184 ered as an appropriate substitute for the stability space of $\mathcal{C}_{N-1}(Q)$. Our
 1185 results show that the former is homotopy equivalent to the classifying space
 1186 of the braid group $\text{Br}(\Gamma_N Q)$, which might be considered as further support
 1187 for this point of view.

1188 **4.3. Discrete derived categories.** This class of triangulated categories
 1189 was introduced and classified by Vossieck [50]; we use the more explicit
 1190 classification in [9]. The contractibility of the stability space, Corollary 4.17
 1191 below, follows from the results of this paper combined with the detailed
 1192 analysis of t-structures on these categories in [18]. [19, Theorem 7.1] provides
 1193 an independent proof of the contractibility of $B\text{Int}(\mathcal{C})$ for a discrete derived
 1194 category \mathcal{C} , using the interpretation of $\text{Int}(\mathcal{C})$ in terms of the poset $\mathbb{P}_2(\mathcal{C})$ of
 1195 silting pairs (Remark 3.14). Combining this with Corollary 3.22 one obtains
 1196 an alternative proof [19, Theorem 8.10] of the contractibility of the stability
 1197 space.

1198 Let A be a finite-dimensional associative algebra over an algebraically-
 1199 closed field. Let $\mathcal{D}(A)$ be the bounded derived category of finite-dimensional
 1200 right A -modules.

1201 **Definition 4.16.** The derived category $\mathcal{D}(A)$ is *discrete* if for each map (of
 1202 sets) $\mu: \mathbb{Z} \rightarrow K(\mathcal{D}(A))$ there are only finitely many isomorphism classes of
 1203 objects $d \in \mathcal{D}(A)$ with $[H^i d] = \mu(i)$ for all $i \in \mathbb{Z}$.

1204 The derived category $\mathcal{D}(Q)$ of a quiver whose underlying graph is an
 1205 ADE Dynkin diagram is discrete. [9, Theorem A] states that if $\mathcal{D}(A)$ is
 1206 discrete but not of this type then it is equivalent as a triangulated category
 1207 to $\mathcal{D}(\Lambda(r, n, m))$ for some $n \geq r \geq 1$ and $m \geq 0$ where $\Lambda(r, n, m)$ is the
 1208 path algebra of the bound quiver in Figure 1. Indeed, $\mathcal{D}(A)$ is discrete if
 1209 and only if A is tilting-cotilting equivalent either to the path algebra of an
 1210 ADE Dynkin quiver or to one of the $\Lambda(r, n, m)$.

1211 Discrete derived categories form an interesting class of examples as they
 1212 are intermediate between the locally-finite case considered in the previous
 1213 section and derived categories of tame representation type algebras. More
 1214 precisely, the distinctions are captured by the Krull–Gabriel dimension of
 1215 the abelianisation, which measures how far the latter is from being a length
 1216 category. In particular, $\text{KGdim}(\text{Ab}(\mathcal{C})) \leq 0$ if and only if \mathcal{C} is locally-finite
 1217 [36]. Krause conjectures [36, Conjecture 4.8] that $\text{KGdim}(\text{Ab}(\mathcal{D}(A))) = 0$
 1218 or 1 if and only if $\mathcal{D}(A)$ is discrete. As evidence he shows that for the full
 1219 subcategory $\text{proj } \mathbf{k}[\epsilon]$ of finitely generated projective modules over the al-
 1220 gebra $\mathbf{k}[\epsilon]$ of dual numbers, $\text{KGdim}(\text{Ab}(\mathcal{D}_b(\text{proj } \mathbf{k}[\epsilon]))) = 1$. The bounded

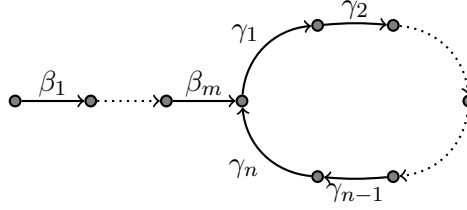


FIGURE 1. The algebra $\Lambda(r, n, m)$ is the path algebra of the quiver $Q(r, n, m)$ above with relations $\gamma_{n-r+1}\gamma_{n-r+2} = \cdots = \gamma_n\gamma_1 = 0$.

1221 derived category $\mathcal{D}(\text{proj } \mathbf{k}[\epsilon])$ is discrete — there are infinitely many inde-
 1222 composable objects, even up to shift, but no continuous families — but
 1223 not locally-finite. Finally, by [24, Theorem 4.3] $\text{KGdim}(\mathcal{D}(A)) = 2$ when
 1224 A is a tame hereditary Artin algebra, for example the path algebra of the
 1225 Kronecker quiver \widehat{A}_1 .

1226 Since the Dynkin case was covered in the previous section we restrict to
 1227 the categories $\mathcal{D}(\Lambda(r, n, m))$. These have finite global dimension if and only
 1228 if $r < n$, and we further restrict to this situation.

1229 **Corollary 4.17** (cf. [19, Theorem 8.10]). *Suppose $\mathcal{C} = \mathcal{D}(\Lambda(r, n, m))$, where*
 1230 *$n > r \geq 1$ and $m \geq 0$. Then the stability space $\text{Stab}(\mathcal{C})$ is contractible.*

1231 *Proof.* By [18, Proposition 6.1] any t-structure on \mathcal{C} is algebraic with only
 1232 finitely many isomorphism classes of indecomposable objects in its heart.
 1233 Lemma 4.1 then shows that each component of the tilting poset has finite-
 1234 type. By Theorem 4.9 $\text{Stab}(\mathcal{C}) = \text{Stab}_{\text{alg}}(\mathcal{C})$, and is a union of contractible
 1235 components. By Lemma 3.17 $\text{Stab}_{\text{alg}}(\mathcal{C})$ is connected. Hence $\text{Stab}(\mathcal{C})$ is
 1236 contractible. \square

1237 **Example 4.18.** The space of stability conditions in the simplest case,
 1238 $(n, r, m) = (2, 1, 0)$, was computed in [52] and shown to be \mathbb{C}^2 . (The category
 1239 was described geometrically in [52], as the constructible derived category
 1240 of \mathbb{P}^1 stratified by a point and its complement, but it is known that in this
 1241 case the constructible derived category is equivalent to the derived category
 1242 of the perverse sheaves, and these have a nearby and vanishing-cycle de-
 1243 scription as representations of the quiver $Q(2, 1, 0)$ with relation $\gamma_2\gamma_1 = 0$.)

1244 5. THE CALABI-YAU- N -CATEGORY OF A DYNKIN QUIVER

1245 **5.1. The category.** In this section we consider in detail another important
 1246 example of a finite-type component, associated to the Ginzburg algebra of an
 1247 ADE Dynkin quiver. We also address the related question of the faithfulness
 1248 of the braid group action on the associated derived category.

1249 Let Q be a quiver whose underlying unoriented graph is an ADE Dynkin
 1250 diagram. Fix $N \geq 2$ and let $\Gamma_N Q$ be the associated Ginzburg algebra of de-
 1251 gree N , let $\mathcal{D}(\Gamma_N Q)$ be the bounded derived category of finite-dimensional
 1252 representations of $\Gamma_N Q$ over an algebraically-closed field \mathbf{k} , and let $\text{Stab}(\Gamma_N Q)$
 1253 be the space of stability conditions on $\mathcal{D}(\Gamma_N Q)$. See [30, §7] for the details
 1254 of the construction of the differential-graded algebra $\Gamma_N Q$ and its derived

category, and for a proof that $\mathcal{D}(\Gamma_N Q)$ is a Calabi–Yau- N category. (Recall that a k -linear triangulated category \mathcal{C} is *Calabi–Yau- N* if, for any objects c, c' in \mathcal{C} we have a natural isomorphism

$$\mathfrak{S}: \mathrm{Hom}_{\mathcal{C}}^{\bullet}(c, c') \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{C}}^{\bullet}(c', c)^{\vee}[N]. \quad (7)$$

Here the graded dual of a graded vector space $V = \bigoplus_{i \in \mathbb{Z}} V_i[i]$ is defined by $V^{\vee} = \bigoplus_{i \in \mathbb{Z}} V_i^*[-i]$. By [1], $\mathrm{Tilt}(\Gamma_N Q)$ and $\mathrm{Stab}(\Gamma_N Q)$ are connected.

Corollary 5.1. *The stability space $\mathrm{Stab}(\Gamma_N Q)$ is of finite-type, and hence is contractible.*

Proof. By [33, Corollary 8.4] each t-structure obtained from the standard one, whose heart is the representations of $\Gamma_N Q$, by a finite sequence of simple tilts is algebraic. [43, Lemma 5.1 and Proposition 5.2] show that each of these t-structures is of finite tilting type. Hence by Lemma 4.1 $\mathrm{Tilt}(\Gamma_N Q)$ has finite tilting type, and therefore by Theorem 4.9 $\mathrm{Stab}(\Gamma_N Q)$ is contractible. \square

This affirms the second part of [43, Conjecture 5.8].

5.2. The braid group. An object s of a k -linear triangulated category is *N -spherical* if $\mathrm{Hom}_{\mathcal{C}}^{\bullet}(s, s) \cong \mathbf{k} \oplus \mathbf{k}[-N]$ and (7) holds functorially for $c = s$ and any c' in \mathcal{C} . The *twist functor* φ_s of a spherical object s was defined in [46] to be

$$\varphi_s(c) = \mathrm{Cone}(s \otimes \mathrm{Hom}^{\bullet}(s, c) \rightarrow c) \quad (8)$$

with inverse $\varphi_s^{-1}(c) = \mathrm{Cone}(c \rightarrow s \otimes \mathrm{Hom}^{\bullet}(s, c)^{\vee})[-1]$. Denote by $\mathcal{D}_{\Gamma Q}$ the canonical heart in $\mathcal{D}(\Gamma_N Q)$, which is equivalent to the module category of Q . Each simple object in $\mathcal{D}_{\Gamma Q}$ is N -spherical cf. [33, § 7.1]. The *braid group* or *spherical twist group* $\mathrm{Br}(\Gamma_N Q)$ of $\mathcal{D}(\Gamma_N Q)$ is the subgroup of $\mathrm{Aut} \mathcal{D}(\Gamma_N Q)$ generated by $\{\varphi_s : s \text{ is simple in } \mathcal{D}_{\Gamma Q}\}$. The lemma below follows directly from the definition of spherical twists.

Lemma 5.2. *Let \mathcal{C} be a k -linear triangulated category, φ_s a spherical twist, and F any auto-equivalence. Then $F \circ \varphi_s = \varphi_{F(s)} \circ F$.*

An important consequence is that two twists φ_s and φ_t by simple objects s and t satisfy the

- braid relation $\varphi_s \varphi_t \varphi_s = \varphi_t \varphi_s \varphi_t$ if and only if $\mathrm{Hom}^{\bullet}(s, t) \cong k[-j]$ for some $j \in \mathbb{Z}$;
- commutativity relation $\varphi_s \varphi_t = \varphi_t \varphi_s$ if and only if $\mathrm{Hom}^{\bullet}(s, t) = 0$;

It follows that there is a surjection

$$\Phi_N: \mathrm{Br}(Q) \twoheadrightarrow \mathrm{Br}(\Gamma_N Q). \quad (9)$$

from the braid group $\mathrm{Br}(Q)$ of the underlying Dynkin diagram, which has a generator b_i for each vertex i and relations $b_i b_j b_i = b_j b_i b_j$ when there is an edge between vertices i and j , and $b_i b_j = b_j b_i$ otherwise. We will show that Φ_N is an isomorphism for any $N \geq 2$. We deal with the cases when $N = 2$, and when Q has type A (for any $N \geq 2$) below; these are already known but we obtain new proofs.

Let \mathfrak{g} be the finite-dimensional complex simple Lie algebra associated to the underlying Dynkin diagram of Q . Let $\mathfrak{h} \subseteq \mathfrak{g}$ denote the Cartan

subalgebra and let $\mathfrak{h}^{\text{reg}} \subseteq \mathfrak{h}$ be the complement of the root hyperplanes in \mathfrak{h} , i.e.

$$\mathfrak{h}^{\text{reg}} = \{\theta \in \mathfrak{h} : \theta(\alpha) \neq 0 \text{ for all } \alpha \in \Lambda\},$$

where Λ is a set of simple roots, i.e. a basis of \mathfrak{h} such that each root can be written as an integral linear combination of basis vectors with either all non-negative or all non-positive coefficients. The Weyl group W is generated by reflections in the root hyperplanes and acts freely on $\mathfrak{h}^{\text{reg}}$.

Theorem 5.3 ([14, Theorem 1.1]). *Let Q be an ADE Dynkin quiver. Then $\text{Stab}(\Gamma_2 Q)$ is a covering space of $\mathfrak{h}^{\text{reg}}/W$ and $\text{Br}(\Gamma_2 Q)$ preserves this component and acts as the group of deck transformations.*

It is well-known that the fundamental group of $\mathfrak{h}^{\text{reg}}/W$ is the braid group $\text{Br}(Q)$ associated to the quiver Q . We therefore obtain new proofs for the following two theorems, by combining Theorem 5.3 and Corollary 5.1.

Theorem 5.4 ([11, Theorem 1.1]). *Let Q be an ADE Dynkin quiver. Then $\Phi_2: \text{Br}(Q) \rightarrow \text{Br}(\Gamma_2 Q)$ is an isomorphism.*

Theorem 5.5 ([20]). *The universal cover of $\mathfrak{h}^{\text{reg}}/W$ is contractible.*

Ikeda has extended Bridgeland–Smith’s work relating stability conditions with quadratic differentials to obtain the following result.

Theorem 5.6 ([26, Theorem 1.1]). *Let Q be a Dynkin quiver of type A . Then there is an isomorphism $\text{Stab}(\Gamma_N Q) / \text{Br}(\Gamma_N Q) \cong \mathfrak{h}^{\text{reg}}/W$ of complex manifolds.*

Combining this with Corollary 5.1, we obtain a new proof of

Theorem 5.7 ([46]). *Let Q be a quiver of type A . Then $\Phi_N: \text{Br}(Q) \rightarrow \text{Br}(\Gamma_N Q)$ is an isomorphism.*

Unfortunately we do not yet know enough about the geometry of the stability spaces for the Calabi–Yau- N categories constructed from Dynkin quivers of other types to deduce the analogous faithfulness of the braid group in those cases. In §6 we give an alternative proof of faithfulness which works for all Dynkin quivers (Corollary 6.14), which also provides a new proof of Theorem 5.5.

Although not phrased in these terms, the above proof is equivalent to showing that the action of $\text{Br}(Q)$ on the combinatorial model $\text{Int}^\circ(\mathcal{D}(\Gamma_N Q))$ of $\text{Stab}(\Gamma_N Q)$ is free. The alternative proof in §6 proceeds by showing instead that the action of $\text{Br}(Q)$ on $\text{Tilt}(\Gamma_N Q)$ is free.

6. THE BRAID ACTION IS FREE

In this section we show that the action of the braid group on $\text{Tilt}(\Gamma_N Q)$ via the surjection $\Phi_N: \text{Br}(Q) \rightarrow \text{Br}(\Gamma_N Q)$ is free. Our strategy uses the isomorphism $\Phi_2: \text{Br}(Q) \rightarrow \text{Br}(\Gamma_2 Q)$ from Theorem 5.6 as a key step, i.e. we bootstrap from the $N = 2$ case. Therefore we assume $N \geq 3$ unless otherwise specified.

For ease of reading we will usually omit Φ_N from our notation when discussing the action, writing simply $b \cdot \mathcal{D}$ for $\Phi_N(b)\mathcal{D}$ where $b \in \text{Br}(Q)$ and $\mathcal{D} \in \text{Tilt}(\Gamma_N Q)$.

1337 **6.1. Local Structure of $\text{Tilt}(\Gamma_N Q)$.** We describe the intervals from \mathcal{D} to
 1338 $L_{\langle s_i, s_j \rangle} \mathcal{D}$ where s_i and s_j are distinct simple objects of the heart of some \mathcal{D} .
 1339 It will be convenient to consider $\text{Tilt}(\Gamma_N Q)$ as a category, with objects the
 1340 elements of the poset and with a unique morphism $\mathcal{D} \rightarrow \mathcal{E}$ whenever $\mathcal{D} \leq \mathcal{E}$.
 1341 The following lemma is the analogue for $\mathcal{D} \in \text{Tilt}(\Gamma_N Q)$ of [43, Lemma 4.3].

Lemma 6.1. *Suppose s_i and s_j are distinct simple objects of the heart of a t -structure $\mathcal{D} \in \text{Tilt}(\Gamma_N Q)$. Then there is either a square or pentagonal commutative diagram of the form*

$$\begin{array}{ccc}
 & L_{s_i} \mathcal{D} & \\
 \nearrow & & \searrow \\
 \mathcal{D} & & L_{\langle s_i, s_j \rangle} \mathcal{D} \\
 \searrow & & \nearrow \\
 & L_{s_j} \mathcal{D} &
 \end{array}
 \quad
 \begin{array}{ccc}
 L_{s_i} \mathcal{D} & \longrightarrow & \mathcal{D}' \\
 \nearrow & & \downarrow \\
 \mathcal{D} & & L_{\langle s_i, s_j \rangle} \mathcal{D} \\
 \searrow & & \nearrow \\
 & L_{s_j} \mathcal{D} & \longrightarrow & L_{\langle s_i, s_j \rangle} \mathcal{D}
 \end{array}
 \tag{10}$$

1342 in $\text{Tilt}(\Gamma_N Q)$, where we may need to exchange i and j to get the precise
 1343 diagram in the pentagonal case, and the t -structure \mathcal{D}' is uniquely specified
 1344 by the diagram. The square occurs when $\text{Hom}^1(s_i, s_j) = 0 = \text{Hom}^1(s_j, s_i)$
 1345 and the pentagon occurs when $\text{Hom}^1(s_i, s_j) = 0$ and $\text{Hom}^1(s_j, s_i) \cong k$.

Proof. First, we claim that either $\text{Hom}^1(s_i, s_j) = 0 = \text{Hom}^1(s_j, s_i)$ or that $\text{Hom}^1(s_i, s_j) = 0$ and $\text{Hom}^1(s_j, s_i) \cong k$. Let the set of simple objects in the heart of \mathcal{D} be $\{s_1, \dots, s_n\}$. By [33, Corollary 8.4 and Proposition 7.4], there is a t -structure \mathcal{E} in $\mathcal{D}(Q)$ such that the Ext-quiver of the heart of \mathcal{D} is the Calabi–Yau- N double of the Ext-quiver of the heart of \mathcal{E} . In other words, one can label the simple objects in the latter as $\{t_1, \dots, t_n\}$ in such a way that

$$\dim \text{Hom}^d(s_k, s_l) = \dim \text{Hom}^d(t_k, t_l) + \dim \text{Hom}^{N-d}(t_l, t_k) \tag{11}$$

1346 for any $1 \leq k, l \leq n$. Moreover, by [43, Lemma 4.2], we have

$$\dim \text{Hom}^\bullet(t_k, t_l) + \dim \text{Hom}^\bullet(t_l, t_k) \leq 1,$$

for any $1 \leq k, l \leq n$. So we may assume, without loss of generality, that $\text{Hom}^\bullet(t_i, t_j) = 0$ and $\text{Hom}^\bullet(t_j, t_i)$ is either zero or is one-dimensional and concentrated in degree d for some $d \in \mathbb{Z}$. Therefore, as $N \geq 3$,

$$\begin{aligned}
 \dim \text{Hom}^1(s_i, s_j) + \dim \text{Hom}^1(s_j, s_i) &= \\
 \dim \text{Hom}^{N-1}(t_j, t_i) + \dim \text{Hom}^1(t_j, t_i) &\leq 1
 \end{aligned}$$

1347 and the claim follows. Since the simple objects $\{s_1, \dots, s_n\}$ are N -spherical,
 1348 and $N \geq 3$, we also note that $\text{Hom}^1(s_i, s_i) = 0 = \text{Hom}^1(s_j, s_j)$ so that
 1349 neither s_i nor s_j has any self-extensions.

1350 The required diagrams arise from the poset of torsion theories in the
 1351 heart of \mathcal{D} which are contained in the extension-closure $\langle s_i, s_j \rangle$. This is
 1352 the same as the poset of torsion theories in the full subcategory $\langle s_i, s_j \rangle$.
 1353 When $\text{Hom}^1(s_i, s_j) = 0 = \text{Hom}^1(s_j, s_i)$ this subcategory is equivalent to
 1354 representations of the quiver with two vertices and no arrows, and when
 1355 $\text{Hom}^1(s_j, s_i) = 0$ and $\text{Hom}^1(s_i, s_j) \cong k$ it is equivalent to representations
 1356 of the A_2 quiver. Identifying torsion theories with the set of non-zero inde-
 1357 composable objects contained within them we have four in the first case —

1358 $\emptyset, \{s_j\}, \{s_i\}$, and $\{s_j, s_i\}$ — and five in the second — $\emptyset, \{s_j\}, \{s_i\}, \{e, s_i\}$,
 1359 and $\{s_j, s_i\}$ where e is the indecomposable extension $0 \rightarrow s_j \rightarrow e \rightarrow s_i \rightarrow 0$.
 1360 These clearly give rise to the square and pentagonal diagrams above. More-
 1361 over, note that $\mathcal{D}' = L_{\langle s_i, e \rangle} \mathcal{D}$ is uniquely specified as claimed. \square

1362 **Remark 6.2.** Recall from Lemma 2.13 that $\text{Tilt}(\Gamma_N Q)$ is a lattice. It follows
 1363 that the above lemma allows us to give a presentation for the category
 1364 $\text{Tilt}(\Gamma_N Q)$ in terms of generating morphisms and relations. The generators
 1365 are the simple left tilts. The relations are provided by the squares and
 1366 pentagons of the above lemma.

1367 **6.2. Associating generating sets.** By [33, Corollary 8.4] the simple ob-
 1368 jects of the heart of any t-structure in $\text{Tilt}(\Gamma_N Q)$ are N -spherical, and the
 1369 associated spherical twists form a generating set for $\text{Br}(\Gamma_N Q)$. Moreover,
 1370 we can explicitly describe how the generating set changes as we perform a
 1371 simple tilt. Let s_1, \dots, s_n be the simple objects of the heart of \mathcal{D} . By [33,
 1372 Proposition 5.4 and Remark 7.1], the simple objects of the heart of $L_{s_i} \mathcal{D}$
 1373 are

$$\{s_i[-1]\} \cup \{s_k : \text{Hom}^1(s_i, s_k) = 0, k \neq i\} \cup \{\varphi_{s_i}(s_j) : \text{Hom}^1(s_i, s_j) \neq 0\}. \quad (12)$$

1374 As $\varphi_{\varphi_{s_i}(s_j)} = \varphi_{s_i} \varphi_{s_j} \varphi_{s_i}^{-1}$ by Lemma 5.2,

$$\{\varphi_{s_i}\} \cup \{\varphi_{s_k} : \text{Hom}^1(s_i, s_k) = 0\} \cup \{\varphi_{s_i} \varphi_{s_j} \varphi_{s_i}^{-1} : \text{Hom}^1(s_i, s_j) \neq 0\} \quad (13)$$

1375 is the new generating set for $\text{Br}(\Gamma_N Q)$. In this section we lift the above
 1376 generating sets, in certain cases, along the surjection Φ_N to generating sets
 1377 for $\text{Br}(Q)$.

Let $\mathcal{D}_{\Gamma Q}$ be the standard t-structure in $\mathcal{D}(\Gamma_N Q)$. By [33, Theorem 8.6]
 there is a canonical bijection

$$\mathcal{I}_{\Gamma_N Q} \xrightarrow{1-1} \text{Tilt}(\Gamma_N Q) / \text{Br}(\Gamma_N Q), \quad (14)$$

where $\mathcal{I}_{\Gamma_N Q}$ is the full subcategory of $\text{Tilt}(\Gamma_N Q)$ consisting of t-structures
 between $\mathcal{D}_{\Gamma Q}$ and $\mathcal{D}_{\Gamma Q}[2-N]$. Let \mathcal{D}_Q be the standard t-structure in $\mathcal{D}(Q)$
 and let \mathcal{I}_Q be the full subcategory of $\text{Tilt}^\circ(Q)$ consisting of t-structures
 between \mathcal{D}_Q and $\mathcal{D}_Q[2-N]$. Recall from [33, Definition 7.3, §8] that there
 is a strong Lagrangian immersion $\mathcal{L}^N : \mathcal{D}(Q) \rightarrow \mathcal{D}(\Gamma_N Q)$, i.e. a triangulated
 functor with the additional property that for any $x, y \in \mathcal{D}(Q)$,

$$\text{Hom}^d(\mathcal{L}^N(x), \mathcal{L}^N(y)) \cong \text{Hom}^d(x, y) \oplus \text{Hom}^{N-d}(y, x)^*. \quad (15)$$

In this case, by [33, Theorem 8.6], the Lagrangian immersion induces an
 isomorphism

$$\mathcal{L}_*^N : \mathcal{I}_Q \rightarrow \mathcal{I}_{\Gamma_N Q}, \quad (16)$$

1378 sending \mathcal{D}_Q to $\mathcal{D}_{\Gamma Q}$. Moreover, for $\mathcal{E} \in \mathcal{I}_Q$ the simple objects of the heart of
 1379 $\mathcal{L}_*^N(\mathcal{E}) \in \mathcal{I}_{\Gamma_N Q}$ are the images under \mathcal{L}^N of the simple objects of the heart
 1380 of \mathcal{E} .

1381 Denote by $\text{Ind} \mathcal{C}$ the set of indecomposable objects in an additive category
 1382 \mathcal{C} . For any acyclic quiver Q , it is known that $\text{Ind} \mathcal{D}(Q) = \bigcup_{l \in \mathbb{Z}} \text{Ind} \mathcal{D}_Q[l]$

where \mathcal{D}_Q is the standard heart. By Theorem 5.4 there is an isomorphism $\Phi_2^{-1}: \text{Br}(\Gamma_2 Q) \rightarrow \text{Br}(Q)$. We define a map

$$b: \text{Ind } \mathcal{D}(Q) \rightarrow \text{Br}(Q) : x \mapsto \Phi_2^{-1}(\varphi_{\mathcal{L}^2(x)}).$$

To spell it out, we first send x to $\mathcal{L}^2(x)$, which is a 2-spherical object in $\mathcal{D}(\Gamma_2 Q)$ (see the lemma below), and then take the image of its spherical twist in $\text{Br}(Q)$ under the isomorphism Φ_2^{-1} . Note that b is invariant under shifts.

Lemma 6.3. *Let $x, y \in \text{Ind } \mathcal{D}(Q)$. Then*

- (1) $\mathcal{L}^2(x)$ is a 2-spherical object for any $x \in \text{Ind } \mathcal{D}(Q)$;
- (2) if $\text{Hom}^\bullet(x, y) = \text{Hom}^\bullet(y, x) = 0$, then $b(x)b(y) = b(y)b(x)$;
- (3) if there is a triangle $y \rightarrow z \rightarrow x \rightarrow y[1]$ in $\text{Ind } \mathcal{D}(Q)$ for some $z \in \text{Ind } \mathcal{D}(Q)$, then $b(z) = b(x)b(y)b(x)^{-1}$ and

$$b(x)b(y)b(x) = b(y)b(x)b(y),$$

i.e. $b(x)$ and $b(y)$ satisfy the braid relation.

Proof. Let x be an indecomposable in $\mathcal{D}(Q)$. Then, by [43, Lemma 2.4], x induces a section $P(x)$ of the Auslander–Reiten quiver of $\mathcal{D}(Q)$, and hence a t-structure $\mathcal{D}_x = [P(x), \infty)$. For a Dynkin quiver, all such t-structures are known to be related to the standard t-structure by tilting, so $\mathcal{D}_x \in \text{Tilt}^\circ(Q)$. Moreover, again by [43, Lemma 2.4], the heart of \mathcal{D}_x is isomorphic to the category of kQ' modules for some quiver Q' with the same underlying diagram as Q . It follows that the section $P(x)$ is isomorphic to $(Q')^{\text{op}}$ and consists of the projective representations of kQ' . By definition x is a source of the section, so is the projective corresponding to a sink in Q' , and is therefore a simple object of the heart. By [33, Corollary 8.4] the image of any such simple object is 2-spherical. Hence (1) follows.

For ease of reading, denote by \tilde{x}, \tilde{y} and \tilde{z} the images of x, y and z respectively under \mathcal{L}^2 . When x and y are orthogonal (15) implies

$$\text{Hom}^\bullet(\tilde{x}, \tilde{y}) = \text{Hom}^\bullet(\tilde{y}, \tilde{x}) = 0,$$

and so the associated twists commute.

To prove (3) note that the triangle $y \rightarrow z \rightarrow x \rightarrow y[1]$ induces a non-trivial triangle in $\mathcal{D}(\Gamma_2 Q)$ via \mathcal{L}^2 . By [43, Lemma 4.2]

$$\text{Hom}^\bullet(x, y) \cong \mathbf{k}[-1] \quad \text{and} \quad \text{Hom}^\bullet(y, x) = 0.$$

Thus (15) yields $\text{Hom}^\bullet(\tilde{x}, \tilde{y}) \cong \mathbf{k}[-1]$ and $\text{Hom}_\bullet^{\tilde{y}}(\tilde{x}, \cong) \mathbf{k}[-1]$, and we deduce that $\tilde{z} = \varphi_{\tilde{x}}(\tilde{y}) = \varphi_{\tilde{y}}^{-1}(\tilde{x})$. Therefore

$$\varphi_{\tilde{x}} \circ \varphi_{\tilde{y}} \circ \varphi_{\tilde{x}}^{-1} = \varphi_{\tilde{z}} = \varphi_{\tilde{y}}^{-1} \circ \varphi_{\tilde{x}} \circ \varphi_{\tilde{y}},$$

as required. \square

Construction 6.4. We associate to any t-structure in $\text{Tilt}^\circ(Q)$ the generating set $\{b(t_1), \dots, b(t_n)\}$ of $\text{Br}(Q)$ where $\{t_1, \dots, t_n\}$ are the simple objects of the heart. The generating set associated to \mathcal{D}_Q is the standard one.

The following proposition gives an alternative inductive construction of these generating sets which we use in the sequel.

Proposition 6.5. *Suppose \mathcal{D} is a t-structure in $\mathcal{I}_Q \subseteq \text{Tilt}^\circ(Q)$. Then*

1420 (i) if x and y are two simple objects in the heart of \mathcal{D} one has

$$\begin{cases} b(x)b(y) = b(y)b(x), & \text{if } \mathrm{Hom}^\bullet(x, y) = \mathrm{Hom}^\bullet(y, x) = 0, \\ b(x)b(y)b(x) = b(y)b(x)b(y), & \text{otherwise.} \end{cases}$$

1421 (ii) if $\{t_i\}$ is the set of simple objects in the heart of \mathcal{D} , the simple objects
1422 of the heart of $L_{t_i}\mathcal{D}$ are

$$\{t_i[-1]\} \cup \{t_k : \mathrm{Hom}^1(t_i, t_k) = 0, k \neq i\} \cup \{\varphi_{t_i}(t_j) : \mathrm{Hom}^1(t_i, t_j) \neq 0\} \quad (17)$$

1423 and the corresponding associated generating set of $\mathrm{Br}(Q)$ is

$$\{b_i\} \cup \{b_k : \mathrm{Hom}^1(t_i, t_k) = 0, k \neq i\} \cup \{b_i b_j b_i^{-1} : \mathrm{Hom}^1(t_i, t_j) \neq 0\}, \quad (18)$$

1424 where $\{b_i := b(t_i)\}$ is the generating set associated to \mathcal{D} .

1425 In particular, any such associated set is indeed a generating set of $\mathrm{Br}(Q)$.

1426 Here in (17) we use the notation $\varphi_a(b) := \mathrm{Cone}(a \otimes \mathrm{Hom}^\bullet(a, b) \rightarrow a)$ even
1427 when a is not a spherical object.

1428 *Proof.* First we note that (17) in (ii) is a special case of [33, Proposition 5.4].
1429 The necessary conditions to apply this proposition follow from [33, Theorem
1430 5.9 and Proposition 6.4].

1431 For (i), if x and y are mutually orthogonal then the commutative relations
1432 follow from (2) of Lemma 6.3. Otherwise, by [43, Lemma 4.2],

$$\mathrm{Hom}^\bullet(x, y) \cong \mathbf{k}[-d] \quad \text{and} \quad \mathrm{Hom}^\bullet(y, x) = 0.$$

1433 for some strictly positive integer d . By (17), after tilting \mathcal{D} with respect to
1434 the simple object x (and its shifts) d times we reach a heart with a simple
1435 object $z = \varphi_x(y)$. In particular, there is a triangle $z \rightarrow x[-d] \rightarrow y \rightarrow z[1]$
1436 in $\mathcal{D}(Q)$ where $z \in \mathrm{Ind} \mathcal{D}(Q)$. The braid relation then follows from (3) of
1437 Lemma 6.3.

1438 Finally, (18) in (ii) follows from a direct calculation. \square

1439 We can use this construction to associate generating sets to t-structures in
1440 $\mathcal{I}_{\Gamma_N Q} \subseteq \mathrm{Tilt}(\Gamma_N Q)$. Let \mathcal{E} be such a t-structure, and $\{s_i\}$ the set of simple
1441 objects of its heart. Then $(\mathcal{L}^N)^{-1}s_i$ is well-defined, and we associate the
1442 generating set $\{b_{s_i} := b((\mathcal{L}^N)^{-1}s_i)\}$ of $\mathrm{Br}(Q)$ to \mathcal{E} .

1443 **Remark 6.6.** This construction only works for $\mathcal{E} \in \mathcal{I}_{\Gamma_N Q}$ because the simple
1444 objects of the hearts of other t-structures need not be in the image of the
1445 Lagrangian immersion. This is the same reason that the isomorphism (16)
1446 cannot be extended to the whole of $\mathrm{Tilt}^\circ(Q)$.

1447 The next result follows immediately from Proposition 6.5.

1448 **Corollary 6.7.** Let $\mathcal{E} \in \mathcal{I}_{\Gamma_N Q}$, and let $\{s_i\}$ be the set of simple objects in
1449 its heart, with corresponding generating set $\{b_{s_i}\}$. Then

$$\begin{cases} b_{s_i} b_{s_j} = b_{s_j} b_{s_i}, & \text{if } \mathrm{Hom}^\bullet(s_i, s_j) = 0, \\ b_{s_i} b_{s_j} b_{s_i} = b_{s_j} b_{s_i} b_{s_j}, & \text{otherwise.} \end{cases}$$

1450 Moreover, the simple objects of the heart of $L_{s_i}\mathcal{E}$ are

$$\{s_i[-1]\} \cup \{s_k : \mathrm{Hom}^1(s_i, s_k) = 0, k \neq i\} \cup \{\varphi_{s_i}(s_j) : \mathrm{Hom}^1(s_i, s_j) \neq 0\} \quad (19)$$

1451 and the corresponding associated generating set is

$$\{b_{s_i}\} \cup \{b_{s_k} : \text{Hom}^1(s_i, s_k) = 0, k \neq i\} \cup \{b_{s_i} b_{s_j} b_{s_i}^{-1} : \text{Hom}^1(s_i, s_j) \neq 0\}. \quad (20)$$

1452 **Lemma 6.8.** *Let s be a simple object in the heart of $\mathcal{E} \in \mathcal{I}_{\Gamma_N Q}$. Then either*
 1453 *$L_s \mathcal{E} \in \mathcal{I}_{\Gamma_N Q}$ or $\varphi_s^{-1} L_s \mathcal{E} \in \mathcal{I}_{\Gamma_N Q}$. The first case occurs if and only if, in*
 1454 *addition, $s \in \mathcal{D}_{\Gamma Q}[3 - N]$.*

1455 *Proof.* By [33, Corollary 8.4] the spherical twist φ_s takes \mathcal{E} to the t-structure
 1456 obtained from it by tilting $N - 1$ times ‘in the direction of s ’, i.e. by tilting
 1457 at $s, s[-1], s[-2], \dots, s[3 - N]$ and finally $s[2 - N]$. The first statement
 1458 then follows from the isomorphism $\mathcal{I}_Q \cong \mathcal{I}_{\Gamma_N Q}$ of [33, Theorem 8.1 and
 1459 Proposition 5.13]. For the second statement note that if $L_s \mathcal{E} \in \mathcal{I}_{\Gamma_N Q}$ then
 1460 $s[-1] \in \mathcal{D}_{\Gamma Q}[2 - N]$, so $s \in \mathcal{D}_{\Gamma Q}[3 - N]$, and conversely if $s \notin \mathcal{D}_{\Gamma Q}[3 - N]$
 1461 then $s[-1] \notin \mathcal{D}_{\Gamma Q}[2 - N]$ which implies $L_s \mathcal{E} \notin \mathcal{I}_{\Gamma_N Q}$. \square

1462 The above lemma justifies the following definition.

1463 **Definition 6.9.** Let \mathcal{P} be the poset whose underlying set is

$$\text{Br}(Q) \times \mathcal{I}_{\Gamma_N Q},$$

1464 and whose relation is generated by $(b, \mathcal{E}) \leq (b', \mathcal{E}')$ if either $b = b'$ and $\mathcal{E} \leq \mathcal{E}'$
 1465 in $\mathcal{I}_{\Gamma_N Q}$, or $b' = b \cdot b_s$ and $\mathcal{E}' = \varphi_s^{-1} L_s \mathcal{E}$ where s is a simple object of the
 1466 heart of \mathcal{E} with the property that $L_s \mathcal{E} \notin \mathcal{I}_{\Gamma_N Q}$, equivalently, by Lemma 6.8,
 1467 $s \notin \mathcal{D}_{\Gamma Q}[3 - N]$.

1468 **Lemma 6.10.** *There is a map of posets*

$$\alpha: \mathcal{P} \rightarrow \text{Tilt}(\Gamma_N Q) : (b, \mathcal{E}) \mapsto b \cdot \mathcal{E} := \Phi_N(b) \mathcal{E},$$

1469 *which is surjective on objects and on morphisms. Moreover, \mathcal{P} is connected*
 1470 *and α is equivariant with respect to the canonical free left $\text{Br}(Q)$ -action on*
 1471 *\mathcal{P} .*

1472 *Proof.* To check that α is a map of posets we need only check that the
 1473 generating relations for \mathcal{P} map to relations in $\text{Tilt}(\Gamma_N Q)$. This is clear since
 1474 (in either case) $b' \cdot \mathcal{E}' = b \cdot L_s \mathcal{E} = L_{b \cdot s}(b \cdot \mathcal{E})$. It is surjective on objects by
 1475 [33, Proposition 8.3]. To see that it is surjective on morphisms it suffices
 1476 to check that each morphism $\mathcal{F} \leq L_t \mathcal{F}$, where t is a simple object of the
 1477 heart of \mathcal{F} , lifts to \mathcal{P} . For this, suppose $\mathcal{F} = b \cdot \mathcal{E}$ where $\mathcal{E} \in \mathcal{I}_{\Gamma_N Q}$, and
 1478 that $t = b \cdot s$ for simple s in the heart of \mathcal{E} . Then either $L_s \mathcal{E} \in \mathcal{I}_{\Gamma_N Q}$ and
 1479 $(b, \mathcal{E}) \leq (b, L_s \mathcal{E})$ is the required lift, or $L_s \mathcal{E} \notin \mathcal{I}_{\Gamma_N Q}$ and

$$(b, \mathcal{E}) \leq (b \cdot b_s, \varphi_s^{-1} L_s \mathcal{E})$$

1480 is the required lift.

1481 The connectivity of \mathcal{P} follows from the facts that $(b, \mathcal{E}) \leq (b \cdot b_s, \mathcal{E})$ for any
 1482 simple object s of the heart of $\mathcal{E} \in \mathcal{I}_{\Gamma_N Q}$ and that $\mathcal{I}_{\Gamma_N Q}$ is connected. Finally,
 1483 the equivariance with respect to the left $\text{Br}(Q)$ -action $b' \cdot (b, \mathcal{E}) = (b'b, \mathcal{E})$ is
 1484 clear. \square

1485 **Proposition 6.11.** *The morphism $\alpha: \mathcal{P} \rightarrow \text{Tilt}(\Gamma_N Q)$ is a covering.*

1486 *Proof.* By Lemma 6.10 we know α is surjective on objects and on morphisms,
 1487 so all we need to show is that each morphism lifts *uniquely* to \mathcal{P} once the
 1488 source is given. By Remark 6.2 it suffices to show that the squares and
 1489 pentagons (10) of Lemma 6.1 lift to \mathcal{P} . Using the $\text{Br}(Q)$ -action on \mathcal{P} it
 1490 suffices to show that the diagrams with source \mathcal{D} lift to diagrams with source
 1491 $(1, \mathcal{D})$. We treat only the case of the pentagon, since the square is similar
 1492 but simpler. We use the notation of Lemma 6.1: s_i and s_j are simple objects
 1493 in the heart of $\mathcal{D} \in \mathcal{I}_{\Gamma_N Q}$ with $\text{Hom}^1(s_i, s_j) \cong k$ and $\text{Hom}^1(s_j, s_i) \cong 0$, and
 1494 e is the extension sitting in the non-trivial triangle $s_j \rightarrow e \rightarrow s_i \rightarrow s_j[1]$.

1495 There are four cases depending on whether or not $L_{s_i} \mathcal{D}$ and $L_{s_j} \mathcal{D}$ are in
 1496 $\mathcal{I}_{\Gamma_N Q}$ or not.

1497 **Case A:** If $L_{s_i} \mathcal{D}, L_{s_j} \mathcal{D} \in \mathcal{I}_{\Gamma_N Q}$ then $L_{\langle s_i, s_j \rangle} \mathcal{D} = L_{s_i} \mathcal{D} \vee L_{s_j} \mathcal{D} \in \mathcal{I}_{\Gamma_N Q}$ too.

1498 Hence there is obviously a lifted diagram in $1 \times \mathcal{I}_{\Gamma_N Q}$.

1499 **Case B:** If $L_{s_i} \mathcal{D} \notin \mathcal{I}_{\Gamma_N Q}$ but $L_{s_j} \mathcal{D} \in \mathcal{I}_{\Gamma_N Q}$ then we claim that

$$\begin{array}{ccc}
 & (b_{s_i}, \varphi_{s_i}^{-1} L_{s_i} \mathcal{D}) & \xrightarrow{\varphi_{s_i}^{-1} e} (b_{s_i}, \varphi_{s_i}^{-1} \mathcal{D}') \\
 \begin{array}{c} s_i \\ \nearrow \\ (1, \mathcal{D}) \\ \searrow \\ s_j \end{array} & & \downarrow \varphi_{s_i}^{-1} s_j \\
 & (1, L_{s_j} \mathcal{D}) & \xrightarrow{s_i} (b_{s_i}, \varphi_{s_i}^{-1} L_{\langle s_i, s_j \rangle} \mathcal{D})
 \end{array}$$

1500 is the required lift. (Here, and in the sequel, we label the morphisms
 1501 by the associated simple object.) To confirm this we note that by
 1502 Lemma 6.8 $s_i \notin \mathcal{D}_{\Gamma Q}[3-N]$, from which it follows that the bottom
 1503 morphism is in \mathcal{P} , and that similarly $\varphi_{s_i}^{-1} e = s_j \in \mathcal{D}_{\Gamma Q}[3-N]$ so that
 1504 the top morphism is in \mathcal{P} . It follows that the right hand morphism
 1505 is in \mathcal{P} too, because $\varphi_{s_i}^{-1} L_{\langle s_i, s_j \rangle} \mathcal{D} \in \mathcal{I}_{\Gamma_N Q}$.

1506 **Case C:** If $L_{s_i} \mathcal{D} \in \mathcal{I}_{\Gamma_N Q}$ but $L_{s_j} \mathcal{D} \notin \mathcal{I}_{\Gamma_N Q}$ then one can verify that

$$\begin{array}{ccc}
 & (1, L_{s_i} \mathcal{D}) & \xrightarrow{e} (1, \mathcal{D}') \\
 \begin{array}{c} s_i \\ \nearrow \\ (1, \mathcal{D}) \\ \searrow \\ s_j \end{array} & & \downarrow s_j \\
 & (b_{s_j}, \varphi_{s_j}^{-1} L_{s_j} \mathcal{D}) & \xrightarrow{\varphi_{s_j}^{-1} s_i} (b_{s_j}, \varphi_{s_j}^{-1} L_{\langle s_i, s_j \rangle} \mathcal{D})
 \end{array}$$

1507 is the required lift when $\varphi_{s_j}^{-1} s_i = e \in \mathcal{D}_{\Gamma Q}[3-N]$. If $e \notin \mathcal{D}_{\Gamma Q}[3-N]$
 1508 then

$$\begin{array}{ccc}
 & (1, L_{s_i} \mathcal{D}) & \xrightarrow{e} (b_e, \varphi_e^{-1} \mathcal{D}') \\
 \begin{array}{c} s_i \\ \nearrow \\ (1, \mathcal{D}) \\ \searrow \\ s_j \end{array} & & \downarrow \varphi_e^{-1} s_j \\
 & (b_{s_j}, \varphi_{s_j}^{-1} L_{s_j} \mathcal{D}) & \xrightarrow{\varphi_{s_j}^{-1} s_i} (b_{s_j} b_e, \varphi_e^{-1} \varphi_{s_j}^{-1} L_{\langle s_i, s_j \rangle} \mathcal{D})
 \end{array}$$

1509 is the required lift. We need only check that the right-hand morphism
 1510 is in \mathcal{P} . For this note that $\varphi_e^{-1} s_j = s_i[-1]$ so that $b_{\varphi_e^{-1} s_j} = b_{s_i}$, and
 1511 that applying (3) of Lemma 6.3 to the triangle $s_i[-1] \rightarrow s_j \rightarrow e \rightarrow s_i$

1512 we have $b_{s_j} = b_e b_{s_i} b_e^{-1}$, or equivalently $b_{s_j} b_e = b_e b_{\varphi_e^{-1} s_j}$. Moreover,
 1513 since

$$\varphi_{\varphi_e^{-1} s_j}^{-1} L_{\varphi_e^{-1} s_j} \varphi_e^{-1} \mathcal{D}' = \varphi_{\varphi_e^{-1} s_j}^{-1} \varphi_e^{-1} L_{s_j} \mathcal{D}' = \varphi_e^{-1} \varphi_{s_j}^{-1} L_{\langle s_i, s_j \rangle} \mathcal{D},$$

1514 and we already know the latter is in $\mathcal{I}_{\Gamma_N Q}$, we see that the right-hand
 1515 morphism is indeed in \mathcal{P} .

1516 **Case D:** If $L_{s_i} \mathcal{D}, L_{s_j} \mathcal{D} \notin \mathcal{I}_{\Gamma_N Q}$ then the lifted pentagon is

$$\begin{array}{ccc} & (b_{s_i}, \varphi_{s_i}^{-1} L_{s_i} \mathcal{D}) & \xrightarrow{\varphi_{s_i}^{-1} e} (b_{s_i} b_{s_j}, \varphi_{s_j}^{-1} \varphi_{s_i}^{-1} \mathcal{D}') \\ & \nearrow s_i & \downarrow \varphi_{s_j}^{-1} \varphi_{s_i}^{-1} s_j \\ (1, \mathcal{D}) & & \\ & \searrow s_j & \\ & (b_{s_j}, \varphi_{s_j}^{-1} L_{s_j} \mathcal{D}) & \xrightarrow{\varphi_{s_j}^{-1} s_i} (b_{s_j} b_e, \varphi_e^{-1} \varphi_{s_j}^{-1} L_{\langle s_i, s_j \rangle} \mathcal{D}) \end{array}$$

1517 The top morphism is in \mathcal{P} because $\varphi_{s_i}^{-1} e = s_j \notin \mathcal{D}_{\Gamma Q}[3-N]$. The
 1518 bottom morphism is in \mathcal{P} because $\varphi_{s_j}^{-1} s_i = e \notin \mathcal{D}_{\Gamma Q}[3-N]$, for if it
 1519 were then s_i would be in $\mathcal{D}_{\Gamma Q}[3-N]$, which is false by assumption.
 1520 It remains to check that the right-hand morphism is in \mathcal{P} . Note that

$$L_{\varphi_{s_j}^{-1} \varphi_{s_i}^{-1} s_j} \varphi_{s_j}^{-1} \varphi_{s_i}^{-1} \mathcal{D}' = \varphi_{s_j}^{-1} \varphi_{s_i}^{-1} L_{s_j} \mathcal{D}' = \varphi_{s_j}^{-1} \varphi_{s_i}^{-1} L_{\langle s_i, s_j \rangle} \mathcal{D}.$$

1521 Therefore, since we already know that $\varphi_e^{-1} \varphi_{s_j}^{-1} L_{\langle s_i, s_j \rangle} \mathcal{D} \in \mathcal{I}_{\Gamma_N Q}$,
 1522 it suffices to show that $b_{s_i} b_{s_j} = b_{s_j} b_e$, since it then follows that
 1523 $\varphi_{s_j}^{-1} \varphi_{s_i}^{-1} = \varphi_e^{-1} \varphi_{s_j}^{-1}$. The required equation is obtained by applying
 1524 (3) of Lemma 6.3 to the triangle $e \rightarrow s_i \rightarrow s_j[1] \rightarrow e[1]$, and recalling
 1525 that b is invariant under shifts. \square

1526 **Corollary 6.12.** *For $N \geq 2$, the map $\alpha: \mathcal{P} \rightarrow \text{Tilt}(\Gamma_N Q)$ is a $\text{Br}(Q)$ -*
 1527 *equivariant isomorphism, and in particular $\text{Br}(Q)$ acts freely on $\text{Tilt}(\Gamma_N Q)$.*
 1528 *The map $\Phi_N: \text{Br}(Q) \rightarrow \text{Br}(\Gamma_N Q)$ is an isomorphism.*

1529 *Proof.* This follows immediately from the fact that $\text{Tilt}(\Gamma_N Q)$ is contractible,
 1530 i.e. has contractible classifying space, and that $\alpha: \mathcal{P} \rightarrow \text{Tilt}(\Gamma_N Q)$ is a
 1531 connected $\text{Br}(Q)$ -equivariant cover on which $\text{Br}(Q)$ acts freely.

1532 Recall that $\text{Br}(Q)$ acts on $\text{Tilt}(\Gamma_N Q)$ via the surjective homomorphism
 1533 Φ_N . Since the action is free Φ_N must also be injective, and therefore is an
 1534 isomorphism. \square

1535 **Remark 6.13.** When Q is of type A, Corollary 6.12 provides a third proof
 1536 of Theorem 5.7. When Q is of type E, it shows that there is a faithful sym-
 1537 plectic representation of the braid group, because $\mathcal{D}(\Gamma_N Q)$ is a subcategory
 1538 of a derived Fukaya category, while the spherical twists are the higher ver-
 1539 sion of Dehn twists. This is contrary to the result in [51] in the surface case,
 1540 which says that there is no faithful geometric representation of the braid
 1541 group of type E.

1542 **Corollary 6.14.** *For $N \geq 2$, the induced action of $\text{Br}(Q)$ on $\text{Stab}(\Gamma_N Q)$ is*
 1543 *free.*

1544 *Proof.* If an element of $\text{Br}(Q)$ fixes $\sigma \in \text{Stab}(\Gamma_N Q)$ then it must fix the
 1545 associated t-structure in $\text{Tilt}(\Gamma_N Q)$. \square

1546 Note that we recover the well-known fact that $\mathrm{Br}(Q)$ is torsion-free from
 1547 this last corollary because $\mathrm{Stab}(\Gamma_N Q)$ is contractible and $\mathrm{Br}(Q)$ acts freely
 1548 so $\mathrm{Stab}(\Gamma_N Q) / \mathrm{Br}(Q)$ is a *finite-dimensional* classifying space for $\mathrm{Br}(Q)$.
 1549 The classifying space of any group with torsion must be infinite-dimensional.

1550 **6.3. Higher cluster theory.** The quotient $\mathrm{Tilt}(\Gamma_N Q) / \mathrm{Br}(Q)$ has a nat-
 1551 ural description in terms of higher cluster theory. We recall the relevant
 1552 notions from [33, Section 4]. As previously, $\mathcal{D}(Q)$ is the bounded derived
 1553 category of the quiver Q .

1554 **Definition 6.15.** For any integer $m \geq 2$, the *m-cluster shift* is the auto-
 1555 equivalence of $\mathcal{D}(Q)$ given by $\Sigma_m = \tau^{-1} \circ [m-1]$, where τ is the Auslander-
 1556 Reiten translation. The *m-cluster category* $\mathcal{C}_m(Q) = \mathcal{D}(Q) / \Sigma_m$ is the orbit
 1557 category, which is Calabi–Yau- m . When it is clear from the context we will
 1558 omit the index m from the notation.

1559 An *m-cluster tilting set* $\{p_j\}_{j=1}^n$ in $\mathcal{C}_m(Q)$ is an Ext-configuration, i.e. a
 1560 maximal collection of non-isomorphic indecomposable objects such that

$$\mathrm{Ext}_{\mathcal{C}_m(Q)}^k(p_i, p_j) = 0, \text{ for } 1 \leq k \leq m-1.$$

1561 Any *m-cluster tilting set* consists of $n = \mathrm{rank} K\mathcal{D}(Q)$ objects.

1562 New cluster tilting sets can be obtained by mutations. The *forward muta-*
 1563 *tion* $\mu_{p_i}^\sharp P$ of an *m-cluster tilting set* $P = \{p_j\}_{j=1}^n$ at the object p_i is obtained
 1564 by replacing p_i by

$$p_i^\sharp = \mathrm{Cone}(p_i \rightarrow \bigoplus_{j \neq i} \mathrm{Irr}(p_i, p_j)^* \otimes p_j).$$

1565 Here $\mathrm{Irr}(p_i, p_j)$ is the space of irreducible maps from p_i to p_j in the full
 1566 additive subcategory $\mathrm{Add}(\bigoplus_{i=1}^n p_i)$ of $\mathcal{C}_m(Q)$ generated by the objects of
 1567 the original cluster tilting set. Similarly, the *backward mutation* $\mu_{p_i}^b P$ is
 1568 obtained by replacing p_i by

$$p_i^b = \mathrm{Cone}(\bigoplus_{j \neq i} \mathrm{Irr}(p_j, p_i) \otimes p_j \rightarrow p_i)[-1].$$

1569 As the names suggest, forward and backward mutation are inverse processes.

1570 Cluster tilting sets in $\mathcal{C}_{N-1}(Q)$ and their mutations are closely related
 1571 to t-structures in $\mathcal{D}(\Gamma_N Q)$ and tilting between them. To be more precise,
 1572 [33, Theorem 8.6], based on the construction of [4, §2], states that $(N-1)$ -
 1573 cluster tilting sets are in bijection with the $\mathrm{Br}(Q)$ -orbits in $\mathrm{Tilt}(\Gamma_N Q)$, and
 1574 that a cluster tilting set P' is obtained from P by a backward mutation if
 1575 and only if each t-structure in the orbit corresponding to P' is obtained by
 1576 a simple left tilt from one in the orbit corresponding to P . This motivates
 1577 the following definition.

Definition 6.16. The *cluster mutation category* $\mathcal{CM}_{N-1}(Q)$ is the category
 whose objects are the $(N-1)$ -cluster tilting sets, and whose morphisms are
 generated by backward mutations subject to the relations that for distinct

$p_i, p_j \in P$ the diagrams

$$\begin{array}{ccc}
 & \mu_{p_i}^b P & \\
 \nearrow & & \searrow \\
 P & & \mu_{p_j}^b \mu_{p_i}^b P \\
 \searrow & & \nearrow \\
 & \mu_{p_j}^b P &
 \end{array}
 \quad
 \begin{array}{ccc}
 \mu_{p_i}^b P & \longrightarrow & \mu_{p_i}^b \mu_{p_j}^b P \\
 \nearrow & & \downarrow \\
 P & & \mu_{p_j}^b \mu_{p_i}^b P \\
 \searrow & & \nearrow \\
 \mu_{p_j}^b P & \longrightarrow & \mu_{p_j}^b \mu_{p_i}^b P
 \end{array}
 \quad (21)$$

commute whenever there is a corresponding lifted diagram of simple left tilts in $\text{Tilt}(\Gamma_N Q)$. Note that, possibly after switching the indices i and j in the pentagonal case, there is always a diagram of one of the above two types.

Proposition 6.17. *There is an isomorphism of categories*

$$\text{Tilt}(\Gamma_N Q) / \text{Br}(Q) \cong \mathcal{CM}_{N-1}(Q).$$

The classifying space of $\mathcal{CM}_{N-1}(Q)$ is a $K(\text{Br}(Q), 1)$.

Proof. The first statement is a rephrasing of [33, Theorem 8.6], using Remark 6.2 and the definition of $\mathcal{CM}_{N-1}(Q)$. The second statement follows from the first and the fact that $\text{Tilt}(\Gamma_N Q)$ is contractible, and the $\text{Br}(Q)$ -action on it free. \square

6.4. Garside groupoid structures. In [34, §1] a Garside groupoid is defined as a group G acting freely on the left of a lattice L in such a way that

- the orbit set $G \backslash L$ is finite;
- there is an automorphism ψ of L which commutes with the G -action;
- for any $l \in L$ the interval $[l, l\psi]$ is finite;
- the relation on L is generated by $l \leq l'$ whenever $l' \in [l, l\psi]$.

The action of $\text{Br}(Q)$ on $\text{Tilt}(\Gamma_N Q)$ provides an example for any $N \geq 3$, in fact a whole family of examples. By Corollary 6.12 the action is free, and by (14) the orbit set is finite. From § 4 we know that $\text{Tilt}(\Gamma_N Q)$ is a lattice, and that closed bounded intervals within it are finite. It remains to specify an automorphism ψ ; we choose $\psi = [-d]$ for any integer $d \geq 1$. It is then clear that the last condition is satisfied since each simple left tilt from \mathcal{D} is in the interval between \mathcal{D} and $\mathcal{D}[-d]$.

In fact the preferred definition of Garside groupoid in [34] is that given in §3, not §1, of that paper. There a Garside groupoid \mathcal{G} is defined to be the groupoid associated to a category \mathcal{G}^+ with a special type of presentation — called a complemented presentation — together with an automorphism $\varphi: \mathcal{G} \rightarrow \mathcal{G}$ (arising from an automorphism of the presentation) and a natural transformation $\Delta: 1 \rightarrow \varphi$ such that

- the category \mathcal{G}^+ is atomic, i.e. for each morphism γ there is some $k \in \mathbb{N}$ such that γ cannot be written as a product of more than k non-identity morphisms;
- the presentation of \mathcal{G} satisfies the cube condition, see [34, §3] for the definition;
- for each $g \in \mathcal{G}^+$ the natural morphism $\Delta_g: g \rightarrow \varphi(g)$ factorises through each generator with source g .

1614 The naturality of Δ is equivalent to the statement that for any generator
 1615 $\gamma: g \rightarrow g'$ we have $\Delta_{g'} \circ \gamma = \varphi(\gamma) \circ \Delta_g$. The collection of data of a comple-
 1616 mented presentation, an automorphism, and a natural transformation sat-
 1617 isfying the above properties is called a *Garside tuple*. See [34, Theorem 3.2]
 1618 for a list of the good properties of a Garside tuple.

1619 Briefly, the translation from the second to the first form of the definition
 1620 is as follows. Fix an object $g \in \mathcal{G}^+$. Let $L = \text{Hom}_{\mathcal{G}}(g, -)$ with the order
 1621 $\gamma \leq \gamma' \iff \gamma^{-1}\gamma' \in \mathcal{G}^+$. Let $G = \text{Hom}_{\mathcal{G}}(g, g)$ acting on L via pre-
 1622 composition. Let the automorphism ψ be given by taking $\gamma: g \rightarrow g'$ to
 1623 $\varphi(\gamma) \circ \Delta_g: g \rightarrow \varphi(g) \rightarrow \varphi(g')$. Note that with these definitions the interval
 1624 $[\gamma, \gamma\psi]$ in the lattice consists of the initial factors of the morphism $\Delta_{g'}$ in
 1625 the category \mathcal{G}^+ .

1626 Below, we verify that cluster mutation category $\mathcal{CM}_{N-1}(Q)$ forms part
 1627 of a Garside tuple.

1628 **Proposition 6.18.** *Let the category \mathcal{G}^+ be $\mathcal{CM}_{N-1}(Q)$, where $N \geq 2$,
 1629 presented as in Definition 6.16. Let the automorphism $\varphi = [-d]$ for an
 1630 integer $d \geq 1$. Let the natural transformation $\Delta_P: P \rightarrow P[-d]$ be given by
 1631 the image under the isomorphism $\text{Tilt}(\Gamma_N Q) / \text{Br}(Q) \cong \mathcal{CM}_{N-1}(Q)$ of the
 1632 unique morphism in $\text{Tilt}(\Gamma_N Q)$ from an object to its shift by $[-d]$. Then
 1633 $(\mathcal{G}^+, \varphi, \Delta)$ is a Garside tuple.*

1634 *Proof.* It is easy to check that the presentation in Definition 6.16 is com-
 1635 plemented — see [34, §3] for the definition. The atomicity of $\mathcal{CM}_{N-1}(Q)$
 1636 follows from the fact that closed bounded intervals in the cover $\text{Tilt}(\Gamma_N Q)$
 1637 are finite, since this implies that any morphism has only finitely many fac-
 1638 torisations into non-identity morphisms. The factorisation property follows
 1639 from the inequalities

$$\mathcal{D} \leq L_s \mathcal{D} \leq \mathcal{D}[-d]$$

1640 for any simple object s of the heart of any t-structure \mathcal{D} . Finally the cube
 1641 condition follows from the fact that the cover $\text{Tilt}(\Gamma_N Q)$ is a lattice. \square

1642 **Remark 6.19.** In the case $N = 3$ and $d = 1$ the natural morphism Δ_P is a
 1643 maximal green mutation sequence, in the sense of Keller (cf. [29] and [41]).
 1644 For $N > 3$ and $d = N - 2$, the natural transformation Δ should be thought
 1645 as the generalised, or higher, green mutation (for Buan–Thomas’s coloured
 1646 quivers, cf. [33, §6]).

1647 Finally we explain the relationship of the above Garside structure to that
 1648 on the braid group $\text{Br}(Q)$ as described in, for example, [11]. Suppose the
 1649 automorphism φ fixes some object $g \in \mathcal{G}$. Let $G = \text{Hom}_{\mathcal{G}}(g, g)$, and de-
 1650 fine the monoid G^+ analogously. Then we claim G^+ is a Garside monoid,
 1651 and G the associated Garside group — the properties of a complemented
 1652 presentation ensure that G^+ is finitely generated by those generators of \mathcal{G}^+
 1653 with source and target g , and also that it is a cancellative monoid; moreover
 1654 G^+ is atomic since \mathcal{G}^+ is; the cube condition ensures that the partial order
 1655 relation defined by divisibility in G^+ is a lattice; and finally the natural
 1656 transformation Δ yields a central element $\Delta_g \in Z(G)$, which plays the rôle
 1657 of Garside element.

1658 As a particular example note that the automorphism $\varphi = [k(2 - N)]$,
 1659 where $k \in \mathbb{N}$, fixes the standard cluster tilting set in $\mathcal{CM}_{N-1}(Q)$. By

1660 Proposition 6.17 the group of automorphisms is $\text{Br}(Q)$, and thus we obtain
 1661 a Garside group structure on $\text{Br}(Q)$. For a suitable choice of k this agrees
 1662 with that described in [11].

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